

TEST FUNCTIONS, SCHUR-AGLER CLASSES AND TRANSFER-FUNCTION REALIZATIONS: THE MATRIX-VALUED SETTING

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ABSTRACT. Given a collection of test functions, one defines the associated Schur-Agler class as the intersection of the contractive multipliers over the collection of all positive kernels for which each test function is a contractive multiplier. We indicate extensions of this framework to the case where the test functions, kernel functions, and Schur-Agler-class functions are allowed to be matrix- or operator-valued. We illustrate the general theory with two examples: (1) the matrix-valued Schur class over a finitely-connected planar domain and (2) the matrix-valued version of the constrained Hardy algebra (bounded analytic functions on the unit disk with derivative at the origin constrained to have zero value). Emphasis is on examples where the matrix-valued version is not obtained as a simple tensoring with \mathbb{C}^N of the scalar-valued version.

1. INTRODUCTION

In honor of the work of Issai Schur (see [34]), it is common nowadays to refer to the class of holomorphic functions s mapping the unit disk \mathbb{D} into the closed unit disk $\overline{\mathbb{D}}$ as the *Schur class* \mathcal{S} . We summarize some of the many characterizations of the Schur class in the following theorem.

Theorem 1.1. *For a given $s: \mathbb{D} \rightarrow \mathbb{C}$, the following are equivalent:*

- (1) $s \in \mathcal{S}$,
- (2) *the de Branges-Rovnyak kernel associated with s is a positive kernel on \mathbb{D} :*

$$K_s(z, w) := \frac{1 - s(z)\overline{s(w)}}{1 - z\overline{w}} \succeq 0. \quad (1.1)$$

- (3) *s has a unitary transfer-function realization, i.e., there is a unitary colligation matrix $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{X} \oplus \mathbb{C} \rightarrow \mathcal{X} \oplus \mathbb{C}$ so that*

$$s(z) = D + zC(I - zA)^{-1}B. \quad (1.2)$$

- (4) *s satisfies the von Neumann inequality: for any strict contraction operator T on a Hilbert space \mathcal{K} , $\|s(T)\| \leq 1$.*

A natural multivariable generalization of the Schur class from this point of view is to consider functions s defined on the polydisk \mathbb{D}^d (where d is a positive integer). It has been known for some time that the von Neumann inequality fails in

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more than two variables, i.e.: if $d > 2$ there is a holomorphic function s on \mathbb{D}^d (even a polynomial) with $\|s\|_{\mathbb{D}^d} \leq 1$ and a commuting d -tuple $T = (T_1, \dots, T_d)$ of strict contraction operators on a Hilbert space \mathcal{K} for which the multivariable von Neumann inequality

$$\|s(T)\| \leq \|s\|_{\mathbb{D}^d} \quad (1.3)$$

fails. Nevertheless, the subclass of those Schur-class functions over \mathbb{D}^d for which (1.3) does hold, now called the *Schur-Agler class*, does have characterizations analogous to those given in Theorem 1.1 for the single-variable case (see [3, 5, 22]). Note that the analogue of condition (4) in Theorem 1.1 is now used as the definition of the Schur-Agler class. We then have the following analogue of Theorem 1.1

Theorem 1.2. *Given $s: \mathbb{D}^d \rightarrow \mathbb{C}$, the following are equivalent.*

- (1) $s \in \mathcal{SA}_d$.
- (2) *There are positive kernels K_1, \dots, K_d on \mathbb{D}^d so that*

$$1 - s(z)\overline{s(w)} = \sum_{k=1}^d (1 - z_k \overline{w_k}) K_k(z, w). \quad (1.4)$$

- (3) *There is a unitary colligation matrix $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}: \mathcal{X} \oplus \mathbb{C} \rightarrow \mathcal{X} \oplus \mathbb{C}$ and a collection $\{P_1, \dots, P_d\}$ of orthogonal projections with $P_i P_j = 0$ for $i \neq j$ and with $\sum_{j=1}^d P_j = I_{\mathcal{X}}$ so that*

$$s(z) = D + C(I - Z(z)A)^{-1}Z(z)B \quad (1.5)$$

where we have set $Z(z) = z_1 P_1 + \dots + z_d P_d$.

In the test-function approach to defining generalized Schur-Agler classes, going back to the unpublished preprint of Agler [2] and developed further in [6, 27, 29, 41], one proceeds as follows. We here describe the scalar-valued function setting, although the paper [27] deals with a more general semigroupoid setting. One replaces the unit disk \mathbb{D} (or unit polydisk \mathbb{D}^d) with a completely general point set Ω and supposes that one is given a collection of \mathbb{C} -valued functions Ψ on Ω (the set of *test functions*) subject to the condition that $\sup_{\psi \in \Psi} |\psi(z)| < 1$ for each $z \in \Omega$. The set Ψ carries with it a natural completely regular topology, namely, the weakest topology with respect to which each of the functions

$$\mathbb{E}(z): \psi \rightarrow \psi(z), \quad z \in \Omega \quad (1.6)$$

is continuous. One then says that a positive kernel k is Ψ -admissible (written as $k \in \mathcal{K}_{\Psi}$) if multiplication by ψ is contractive as an operator on the reproducing kernel Hilbert space $\mathcal{H}(k)$ associated with k , i.e., if the kernel $K_{\psi,k}(z, w) = (1 - \psi(z)\overline{\psi(w)})k(z, w)$ is positive for each $\psi \in \Psi$. We then say that the function $s: \Omega \rightarrow \mathbb{C}$ is in the Ψ -Schur-Agler class \mathcal{SA}_{Ψ} if multiplication by s is contractive on $\mathcal{H}(k)$ for each $k \in \mathcal{K}_{\Psi}$, i.e., if the kernel $K_{s,k}(z, w) = (1 - s(z)\overline{s(w)})k(z, w)$ is a positive kernel for each $k \in \mathcal{K}_{\Psi}$. We mention that the choice

$$\Omega = \mathbb{D}, \quad \Psi = \{\psi_0(z) = z\} \quad (1.7)$$

leads to the classical Schur class while the choice

$$\Omega = \mathbb{D}^d, \quad \Psi = \{\psi_k(z) = z_k: k = 1, \dots, d\} \quad (1.8)$$

(where $z = (z_1, \dots, z_d) \in \mathbb{D}^d$) leads to the classical Schur-Agler class \mathcal{SA}_d .

The following is the main result concerning the Schur-Agler class \mathcal{SA}_{Ψ} associated with a general test-function collection Ψ .

Theorem 1.3. (See [27, 29] and [8] for an early version.) Given a function $s: \Omega \rightarrow \mathbb{C}$, the following are equivalent.

- (1) $s \in \mathcal{SA}_\Psi$.
- (2) There is a measure ν on Ψ_β (the Stone-Čech compactification of Ψ) and a measurable family $\{K_\psi: \psi \in \Psi_\beta\}$ of positive kernels on Ψ_β so that

$$1 - s(z)\overline{s(w)} = \int_{\Psi_\beta} \left(1 - \psi(z)\overline{\psi(w)}\right) K_\psi(z, w) d\nu(\psi). \quad (1.9)$$

- (3) There is a $C(\Psi_\beta)$ -unitary colligation, i.e., a block unitary operator $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}: \mathcal{X} \oplus \mathbb{C} \rightarrow \mathcal{X} \oplus \mathbb{C}$ together with a $*$ -representation ρ of the C^* -algebra $C(\Psi_\beta)$ (continuous complex-valued functions on Ψ_β) into $\mathcal{L}(\mathcal{X})$ (bounded linear operators on \mathcal{X}), so that

$$s(z) = D + C(I - \rho(\mathbb{E}(z))A)^{-1}\rho(\mathbb{E}(z))B \quad (1.10)$$

(where $\mathbb{E}(z)$ is as in (1.6)).

Note that conditions (2) and (3) in Theorem 1.3 become conditions (2) and (3) in Theorem 1.1 when Ω and Ψ are chosen as in (1.7), and conditions (2) and (3) in Theorem 1.2 when Ω and Ψ are chosen as in (1.8).

A different type of extension of the classical Schur class over the unit disk is the Schur-class $\mathcal{S}_\mathcal{R}$ over a bounded, finitely connected planar domain \mathcal{R} . Here \mathcal{R} is a bounded domain in the complex plane with boundary consisting of $m+1$ disjoint smooth Jordan curves $\partial_0, \partial_1, \dots, \partial_m$, where ∂_0 denotes the boundary of the unbounded component of the complement of \mathcal{R} , and we define $\mathcal{S}_\mathcal{R}$ as the class of all holomorphic functions from \mathcal{R} into the closed disk \mathbb{D}^- . Work in [27, 29] identifies the Schur class $\mathcal{S}_\mathcal{R}$ over \mathcal{R} as a test-function Schur-Agler class $\mathcal{SA}_{\Psi_\mathcal{R}}$ for a certain collection of test functions $\Psi_\mathcal{R} = \{\psi_\mathbf{x}: \mathbf{x} \in \mathbb{T}_\mathcal{R}\}$ indexed by the so-called \mathcal{R} -torus $\mathbb{T}_\mathcal{R}$ defined as the Cartesian product of the connected components of $\partial\mathcal{R}$:

$$\mathbf{x} \in \mathbb{T}_\mathcal{R} := \partial_0 \times \partial_1 \times \dots \times \partial_m.$$

(see Section 4.1 below for complete details). In particular, the decomposition (1.9) in Theorem 1.3 for this case gives us the following: *given $s \in \mathcal{S}_\mathcal{R}$, there is a measure ν on $\mathbb{T}_\mathcal{R}$ and a family of positive kernels $\{k_\mathbf{x}: \mathbf{x} \in \mathbb{T}_\mathcal{R}\}$ so that*

$$1 - s(z)\overline{s(w)} = \int_{\mathbb{T}_\mathcal{R}} \left(1 - \psi_\mathbf{x}(z)\overline{\psi_\mathbf{x}(w)}\right) k_\mathbf{x}(z, w) d\nu(\mathbf{x}). \quad (1.11)$$

We shall be interested in matrix- and operator-valued versions of these Schur and Schur-Agler classes. The operator-valued version of the Schur class over \mathcal{R} , which we denote as $\mathcal{S}_\mathcal{R}(\mathcal{U}, \mathcal{V})$, consists of holomorphic functions S on \mathcal{R} with values $S(z)$ equal to contraction operators between two Hilbert spaces \mathcal{U} and \mathcal{V} . For the case where $\mathcal{R} = \mathbb{D}$, we drop the subscript \mathcal{R} and write simply $\mathcal{S}(\mathcal{U}, \mathcal{V})$; we also abbreviate $\mathcal{S}_\mathcal{R}(\mathcal{U}, \mathcal{U})$ to $\mathcal{S}_\mathcal{R}(\mathcal{U})$. There is also an operator-valued version of the Schur-Agler class over \mathbb{D}^d , namely: *$S: \mathbb{D}^d \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{V})$ is in the Schur-Agler class $\mathcal{SA}_d(\mathcal{U}, \mathcal{V})$ if S is a holomorphic map from \mathbb{D}^d into $\mathcal{L}(\mathcal{U}, \mathcal{V})$ such that $\|S(T)\| \leq 1$ for any commutative tuple $T = (T_1, \dots, T_d)$ of strictly contractive operators on a Hilbert space \mathcal{K} , where we use a tensor functional calculus to define $S(T)$:*

$$S(T) = \sum_{n \in \mathbb{Z}_+^d} S_n \otimes T^n \text{ if } S(z) = \sum_{n \in \mathbb{Z}^d} S_n z^n$$

where we use standard multivariable notation:

$$z^n = z_1^{n_1} \cdots z_d^{n_d}, \quad T^n = T_1^{n_1} \cdots T_d^{n_d} \text{ for } n = (n_1, \dots, n_d) \in \mathbb{Z}_+^d.$$

Then Theorems 1.1 and 1.2 have seamless extensions to the matrix-/operator-valued settings. Indeed, $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ if and only if the de Branges-Rovnyak $\mathcal{L}(\mathcal{Y})$ -valued kernel

$$K_S(z, w) := \frac{I_{\mathcal{Y}} - S(z)S(w)^*}{1 - z\bar{w}}$$

is a positive kernel on \mathbb{D} if and only if there is a unitary colligation matrix $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{X} \oplus \mathcal{U} \rightarrow \mathcal{X} \oplus \mathcal{Y}$ so that $S(z) = D + zC(I - zA)^{-1}B$. Similarly, $S \in \mathcal{SA}_d(\mathcal{U}, \mathcal{Y})$ if and only if there are positive $\mathcal{L}(\mathcal{E})$ -valued kernels K_1, \dots, K_d on \mathbb{D}^d so that $I - S(z)S(w)^* = \sum_{k=1}^d (1 - z_k \bar{w}_k) K_k(z, w)$ if and only if S has a representation as in (1.5) but with \mathbf{U} acting from $\mathcal{X} \oplus \mathcal{U}$ to $\mathcal{X} \oplus \mathcal{Y}$. We mention that this result has inspired several variants where the polydisk \mathbb{D}^d is replaced by a more general domain \mathcal{D}_Q in \mathbb{C}^d specified by a polynomial (or more generally analytic) matrix-valued determining function Q : $\mathcal{D}_Q = \{z \in \mathbb{C}^d : \|Q(z)\| < 1\}$; more generally the technique of the proof going through the transfer-function realization naturally leads to interpolation and commutant lifting versions of the result (see [22, 21, 53, 23, 10, 16, 9]). We mention that there is now also a noncommutative version of the Schur-Agler class [19].

However, for the case $\mathcal{S}_{\mathcal{R}}(\mathbb{C}^N)$, the expected matrix generalization of (1.11), namely

$$I - S(z)S(w)^* = \int_{\mathbb{T}_{\mathcal{R}}} \left(1 - \psi_{\mathbf{x}}(z)\overline{\psi_{\mathbf{x}}(w)}\right) K_{\mathbf{x}}(z, w) \, d\nu(\mathbf{x}) \quad (1.12)$$

for a measurable family $\{K_{\mathbf{x}} : \mathbf{x} \in \mathbb{T}_{\mathcal{R}}\}$ of positive $N \times N$ matrix-valued kernels on \mathcal{R} , fails in general, at least in the case where \mathcal{R} is a region with three holes having some additional symmetry properties; indeed this phenomenon is a key ingredient in the negative answer to the spectral set question for such regions \mathcal{R} obtained by Dritschel and McCullough in [28].

One of the main motivations for the present paper is to develop a framework of test-function Schur-Agler class \mathcal{SA}_{Ψ} for the case of matrix- or operator-valued test functions Ψ and to recover a formula of the type (1.12) for the Schur class $\mathcal{S}_{\mathcal{R}}(\mathbb{C}^N)$ for an appropriately enlarged class $\Psi_{\mathcal{R}}^N$ of matrix-valued test functions. We therefore develop a systematic extension of the work of [27, 29] to the matrix- and operator-valued setting: this is the main content of Section 3 below. We also emphasize the interpolation version of the main result, whereby one characterizes which functions S_0 defined on some subset Ω_0 of Ω can be extended to a test-function Schur-Agler-class function S defined on all of Ω . Most of the analysis builds on the earlier work of [3, 5, 22, 10, 16, 8, 27, 29], but there are places where new ideas and techniques were required.

In Section 4 we take two algebras which are intrinsically defined and identify their unit balls as also arising as test-function Schur-Agler classes. The first has already been mentioned: namely, the algebra of bounded holomorphic $N \times N$ matrix functions over a multiply-connected planar domain \mathcal{R} whose unit ball is the Schur class $\mathcal{S}_{\mathcal{R}}(\mathbb{C}^N)$. The second is the matrix-valued version of the constrained Hardy algebra over the unit disk \mathbb{D} (bounded holomorphic functions f on \mathbb{D} subject to the constraint that $f'(0) = 0$). The first example has been an object of much study over the years (see [1, 14, 18, 4, 28, 54]) while interest in the second is more recent [26, 17, 50]. Motivation for study of the second algebra comes from the fact that

it is a model for the bounded analytic functions on the intersection of a variety V embedded in \mathbb{C}^2 with the unit bidisk (see [7]). For these two examples we identify an appropriate class of test functions Ψ^N so that the unit ball of the given algebra is equal to the matrix-valued test-function Schur-Agler class \mathcal{SA}_{Ψ^N} associated with Ψ^N . It is always possible to choose Ψ^N simply as the unit ball of the given algebra; the point is to find a valid class Ψ^N which is as small as possible. As has already been mentioned for the first example, in both examples the test-function class Ψ^1 identified in previous work ([29, 30]) for the scalar-valued version fails to work for the matrix-valued case. For each of these two examples, we find a valid test-function class Ψ^N as a linear-fractional transform of the set of extreme points of a normalized matrix-valued Herglotz (positive real part) version of the algebra, just as has been done for the scalar-valued case in [28, 28, 30]. Identification of these extreme points for the matrix-valued case leads us to draw on results from [20] concerning extreme points for a convex cone of matrix quantum probability measures (positive matrix-valued measures with total mass equal to the identity matrix). The resulting test-function classes are not as explicit as in the scalar-valued settings; however, for the Schur class $\mathcal{S}_{\mathcal{R}}$ with \mathcal{R} equal to an annulus, we are able to use results of McCullough [38] to obtain a more explicit test-function class and use the resulting matrix-valued continuous Agler decomposition (the matrix-valued analogue of (1.9)) to obtain a variant of McCullough's positive solution of the spectral set question for an annulus.

A criticism of the study of Schur-Agler classes in general is that their intrinsic structure is a priori mysterious: after going through the several steps of the definition, one does not have any intrinsic characterization of the eventual result. Our work in Section 4 (as well as the work in [29, 30]) counterbalances this concern by starting with an intrinsically defined function algebra and identifying it as a Schur-Agler class. There are now papers obtaining characterizations of which operator algebras have unit balls equal to a Schur-Agler class (see [42, 36]). Other work [37] characterizes families of kernels so that the associated contractive multipliers form a test-function Schur-Agler class. It should be of interest to extend these results to the matrix-valued setting in the spirit of the present paper.

The paper is organized as follows. Section 2 presents some preliminary material on test functions, positive kernels, and structured unitary colligation matrices needed in the sequel. Section 3 presents the main structure result (including the interpolation version as well as a representation-theoretic version) for the general matrix-valued test-function Schur-Agler class. Section 4 develops the two illustrative examples of matrix-valued Schur classes which can be identified as test-function Schur-Agler classes. Finally we mention that this paper together with [20] form an enhanced version of the second author's dissertation [35].

2. PRELIMINARIES

2.1. Test functions. We assume that we are given two coefficient Hilbert spaces \mathcal{U}_T and \mathcal{Y}_T and a collection Ψ of functions ψ on the abstract set of points Ω with values in the space $\mathcal{L}(\mathcal{U}_T, \mathcal{Y}_T)$ of bounded linear operators between \mathcal{U}_T and \mathcal{Y}_T . We say that Ψ is a *collection of test functions* if it happens that

$$\sup\{\|\psi(z)\| : \psi \in \Psi\} < 1 \text{ for each } z \in \Omega. \quad (2.1)$$

We view Ψ as a subset of $B(\Omega, \overline{\mathcal{BL}}(\mathcal{U}_T, \mathcal{Y}_T))$ (the space of (bounded) maps from Ω into the closed unit ball of bounded linear operators between \mathcal{U}_T and \mathcal{Y}_T). We topologize $B(\Omega, \overline{\mathcal{BL}}(\mathcal{U}_T, \mathcal{Y}_T))$ with the topology of pointwise weak-* convergence, i.e., we view $B(\Omega, \overline{\mathcal{BL}}(\mathcal{U}_T, \mathcal{Y}_T))$ as the Cartesian product $\Pi_\Omega \overline{\mathcal{BL}}(\mathcal{U}_T, \mathcal{Y}_T)$ with the standard Cartesian product topology). As such $B(\Omega, \overline{\mathcal{BL}}(\mathcal{U}_T, \mathcal{Y}_T))$ is compact by Tychonoff's Theorem ([31, Theorem XI.1.4]), since each fiber $\overline{\mathcal{BL}}(\mathcal{U}_T, \mathcal{Y}_T)$ is compact by the Banach-Alaoglu Theorem [51, Theorem 3.15]. As a subspace of the completely regular space $B(\Omega, \overline{\mathcal{BL}}(\mathcal{U}_T, \mathcal{Y}_T))$ (i.e., $B(\Omega, \overline{\mathcal{BL}}(\mathcal{U}_T, \mathcal{Y}_T))$ is Hausdorff and any closed set can be separated from a point disjoint from it by a continuous function), Ψ is completely regular in the subspace topology inherited from $B(\Omega, \overline{\mathcal{BL}}(\mathcal{U}_T, \mathcal{Y}_T))$. The closure of Ψ in this topology is compact; however we shall be more interested in the Stone-Čech compactification Ψ_β of Ψ [31, Section XI.8]. Then the space $C_b(\Psi, \mathcal{L}(\mathcal{H}, \mathcal{K}))$ of bounded continuous functions f from Ψ into a space $\mathcal{L}(\mathcal{H}, \mathcal{K})$ of bounded linear operators between two Hilbert spaces \mathcal{H} and \mathcal{K} can be identified with the space $C(\Psi_\beta, \mathcal{L}(\mathcal{H}, \mathcal{K}))$ of continuous functions from the Stone-Čech compactification Ψ_β into $\mathcal{L}(\mathcal{H}, \mathcal{K})$. An operator-valued version of the Riesz representation theorem allows us to identify the dual of $C_b(\Psi, \mathcal{L}(\mathcal{H}, \mathcal{K}))$ with regular, bounded, weakly countably additive $\mathcal{C}_1(\mathcal{K}, \mathcal{H})$ -valued measures on Ψ_β , where we use the notation $\mathcal{C}_1(\mathcal{K}, \mathcal{H})$ to denote the trace-class operators from \mathcal{K} to \mathcal{H} . We note that there are continuous linear functionals L in $C(\Psi_\beta, \mathcal{L}(\mathcal{H}, \mathcal{K}))$ such that allowing points of $\Psi_\beta \setminus \Psi$ to be part of the support of the corresponding measure μ_L is essential (see [29, Section 5.2]).

For each $\psi \in \Psi$ we define the map $\mathbf{ev}_\psi: C_b(\Psi, \mathcal{L}(\mathcal{H}, \mathcal{K})) \rightarrow \mathcal{L}(\mathcal{K})$ by $\mathbf{ev}_\psi: f \rightarrow f(\psi)$. A particular element of $C_b(\Psi, \mathcal{L}(\mathcal{U}_T, \mathcal{Y}_T))$ which will often come up is the function $\mathbb{E}(z)$ (for each $z \in \Omega$) given by

$$\mathbf{ev}_\psi(\mathbb{E}(z)) = \mathbb{E}(z)(\psi) := \psi(z). \quad (2.2)$$

2.2. Positive operator-valued kernels and their multipliers. Let \mathcal{E} be any Hilbert space and suppose that K is a function on $\Omega \times \Omega$ with values in $\mathcal{L}(\mathcal{E})$. We say that K is a *positive kernel* if the Aronszajn condition

$$\sum_{i,j=1}^N \langle K(z_i, z_j) e_j, e_i \rangle_{\mathcal{E}} \geq 0 \text{ for all } z_1, \dots, z_N \in \Omega, e_1, \dots, e_N \in \mathcal{E}, N = 1, 2, \dots \quad (2.3)$$

The following equivalent versions of the positive-kernel condition are often used in function-theoretic operator theory settings.

Theorem 2.1. (See e.g. [6].) Suppose that we are given a function $K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{E})$. Then the following are equivalent:

- (1) K is a positive kernel, i.e., condition (2.3) holds.
- (2) There is a Hilbert space $\mathcal{H}(K)$ consisting of \mathcal{E} -valued functions f such that $K(\cdot, w)e \in \mathcal{H}(K)$ for each $w \in \Omega$ and $e \in \mathcal{E}$ and has the reproducing property:

$$\langle f, K(\cdot, w)e \rangle_{\mathcal{H}(K)} = \langle f(w), e \rangle_{\mathcal{E}} \text{ for all } f \in \mathcal{H}(K).$$

- (3) K has a Kolmogorov decomposition: there is an auxiliary Hilbert space \mathcal{X} and a function $H: \mathcal{X} \rightarrow \mathcal{E}$ so that

$$K(z, w) = H(z)H(w)^*. \quad (2.4)$$

In fact one can take \mathcal{X} to be the reproducing kernel Hilbert space $\mathcal{H}(K)$ described in (2) above with $H(z) = \mathbf{ev}_z: f \mapsto f(z)$.

Rather than using a positive kernel to construct a reproducing kernel Hilbert space as in condition (2) in Theorem 2.1, it is also possible to construct a reproducing kernel Hilbert module as follows. By a Hilbert module over a C^* -algebra \mathfrak{B} we mean a linear space E which is a right module over \mathfrak{B} which is also equipped with an \mathfrak{B} -valued inner product and satisfies additional compatibility requirements with respect to the algebra structure of \mathfrak{B} (see [49, Section 2.1]):

$$\langle \cdot, \cdot \rangle_E: E \times E \rightarrow \mathfrak{B}$$

which satisfies the usual inner product axioms:

- (1) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$,
- (2) $\langle x \cdot b, y \rangle = \langle x, y \rangle b$,
- (3) $\langle x, y \rangle^* = \langle y, x \rangle$,
- (4) $\langle x, x \rangle \geq 0$ (as an element of \mathfrak{B}),
- (5) $\langle x, x \rangle = 0$ implies that $x = 0$,
- (6) E is complete in the norm given by $\|x\| = \|\langle x, x \rangle\|_{\mathfrak{A}}^{1/2}$

for all $x, y, z \in E$, $b \in \mathfrak{B}$ and $\lambda, \mu \in \mathbb{C}$. (Here we follow the mathematicians' (rather than the physicists') convention that inner products are linear in the left slot; this departs from the standard usage in the operator-algebra literature.) By modifying the construction of $\mathcal{H}(K)$ in Theorem 2.1, one can construct a C^* -module, denoted as $\mathcal{H}(K)$, over the C^* -algebra $\mathcal{L}(\mathcal{E})$ characterized as follows.

Theorem 2.2. *Suppose that $K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{E})$ is a positive kernel as in (2.3). Then there is a uniquely determined C^* -module $\mathcal{H}(K)$ over $\mathfrak{B} = \mathcal{L}(\mathcal{E})$ with the following properties:*

- (1) $\mathcal{H}(K)$ consists of $\mathcal{L}(\mathcal{E})$ -valued functions on Ω ,
- (2) for each $w \in \Omega$, $K(\cdot, w)$ is in $\mathcal{H}(K)$ and the span of such elements is dense in $\mathcal{H}(K)$, and
- (3) for each $F \in \mathcal{H}(K)$,

$$\langle F, K(\cdot, w) \rangle_{\mathcal{H}(K)} = F(w) \in \mathcal{L}(\mathcal{E}).$$

Proof. Define an inner product on a pair of kernel elements $K(\cdot, w)$ and $K(\cdot, z)$ by

$$\langle K(\cdot, w), K(\cdot, z) \rangle_{\mathcal{H}(K)} = K(z, w)$$

and extend by linearity to the space of kernel elements. Mod out by any linear combinations having zero self inner product and take the completion to arrive at the space $\mathcal{H}(K)$ having all the asserted properties. Note that there is a version of the Cauchy-Schwarz inequality available (see [49, Lemma 2.5]) which guarantees that the point evaluation map $\mathbf{ev}: f \mapsto f(w)$ extends to elements of the completion, and hence elements of the completion can also be identified as $\mathcal{L}(\mathcal{E})$ -valued functions on Ω . \square

It is natural now to take the next step and introduce the notion of C^* -correspondence (see [43]). Given two C^* -algebras \mathfrak{A} and \mathfrak{B} , by an $(\mathfrak{A}, \mathfrak{B})$ -correspondence we mean a Hilbert module E over \mathfrak{B} which also carries a left \mathfrak{A} -action $x \mapsto a \cdot x$ which is a $*$ -representation of \mathfrak{A} with respect to the \mathfrak{B} -valued inner product on E :

$$\langle a \cdot x, y \rangle_E = \langle x, a^* \cdot y \rangle_E.$$

Given three C^* -algebras \mathfrak{A} , \mathfrak{B} and \mathfrak{C} together with an $(\mathfrak{A}, \mathfrak{B})$ -correspondence E and a $(\mathfrak{B}, \mathfrak{C})$ -correspondence F , the internal tensor product $E \otimes F$ of E and F is defined to be the $(\mathfrak{A}, \mathfrak{C})$ -correspondence generated as the Hausdorff completion of the span of pure tensors $e \otimes f$ ($e \in E$ and $f \in F$) in the \mathfrak{C} -valued inner product given by

$$\langle e \otimes f, e' \otimes f' \rangle_{E \otimes F} = \langle (\langle e, e' \rangle_E) \cdot f, f' \rangle_F \quad (2.5)$$

with left \mathfrak{A} -action given by

$$a \cdot (e \otimes f) = (a \cdot e) \otimes f. \quad (2.6)$$

It is routine to verify that one then gets the balancing property

$$e \otimes (b \cdot f) = (e \cdot b) \otimes f \quad (2.7)$$

for $e \in E$, $f \in F$ and $b \in \mathfrak{B}$.

We shall need a couple of applications of this internal tensor-product construction. The first is as follows. For K an $\mathcal{L}(\mathcal{E})$ -valued positive kernel on Ω , we view the C^* -module over \mathfrak{B} constructed in Theorem 2.2 as a $(\mathbb{C}, \mathcal{L}(\mathcal{E}))$ -correspondence. For \mathcal{X} another coefficient Hilbert space, let $\mathcal{C}_2(\mathcal{X}, \mathcal{E})$ be the space of Hilbert-Schmidt class operators from \mathcal{X} into \mathcal{E} . Then $\mathcal{C}_2(\mathcal{X}, \mathcal{E})$ has a standard Hilbert-space inner product

$$\langle T, T' \rangle_{\mathcal{C}_2(\mathcal{X}, \mathcal{E})} = \text{tr}(TT'^*).$$

We also have a left action of the C^* -algebra $\mathcal{L}(\mathcal{E})$ on $\mathcal{C}_2(\mathcal{X}, \mathcal{E})$ via left multiplication:

$$X \cdot T = XT \text{ for } X \in \mathcal{L}(\mathcal{E}), T \in \mathcal{C}_2(\mathcal{X}, \mathcal{E})$$

and this action gives rise to a $*$ -representation of $\mathcal{L}(\mathcal{E})$ on $\mathcal{C}_2(\mathcal{X}, \mathcal{E})$:

$$\begin{aligned} \langle X \cdot T, T' \rangle_{\mathcal{C}_2(\mathcal{X}, \mathcal{E})} &= \langle XT, T' \rangle_{\mathcal{C}_2(\mathcal{X}, \mathcal{E})} = \text{tr}(XTT'^*) = \text{tr}(TT'^*X) \\ &= \text{tr}(T(X^*T')^*) = \langle T, X^* \cdot T' \rangle_{\mathcal{C}_2(\mathcal{X}, \mathcal{E})}. \end{aligned}$$

In this way we may view $\mathcal{C}_2(\mathcal{X}, \mathcal{E})$ as an $(\mathcal{L}(\mathcal{E}), \mathbb{C})$ -correspondence. We may then form the internal C^* -correspondence tensor-product $\mathcal{H}(K) \otimes \mathcal{C}_2(\mathcal{X}, \mathcal{E})$. Explicitly, the inner product on pure tensors $F \otimes T$ ($F \in \mathcal{H}(K)$, $T \in \mathcal{C}_2(\mathcal{X}, \mathcal{E})$) is given by

$$\langle F \otimes T, F' \otimes T' \rangle_{\mathcal{H}(K) \otimes \mathcal{C}_2(\mathcal{X}, \mathcal{E})} = \text{tr}(\langle F, F' \rangle_{\mathcal{H}(K)} TT'^*).$$

When we evaluate the first factor F in a pure tensor $F \otimes T$ at a point w in Ω , we get a tensor of the form

$$F(w) \otimes T \in \mathcal{L}(\mathcal{E}) \otimes \mathcal{C}_2(\mathcal{X}, \mathcal{E}) \cong \mathcal{C}_2(\mathcal{X}, \mathcal{E}).$$

To interpret this tensor product as a C^* -correspondence internal tensor product, we view $\mathcal{L}(\mathcal{E})$ as a $(\mathcal{L}(\mathcal{E}), \mathcal{L}(\mathcal{E}))$ -correspondence with inner product $\langle X, X' \rangle = X'^*X \in \mathcal{L}(\mathcal{E})$ and left action given by left multiplication: $X' \cdot X = X'X$. The balancing property (2.7) then leads to the identification $\mathcal{L}(\mathcal{E}) \otimes \mathcal{C}_2(\mathcal{X}, \mathcal{E}) \cong \mathcal{C}_2(\mathcal{X}, \mathcal{E})$.

Using a linearity and approximation argument, one can show that in fact elements H of $\mathcal{H}(K) \otimes \mathcal{C}_2(\mathcal{X}, \mathcal{E})$ can be viewed as $\mathcal{C}_2(\mathcal{X}, \mathcal{E})$ -valued functions on Ω such that $K(\cdot, w)U \in \mathcal{H}(K) \otimes \mathcal{C}_2(\mathcal{X}, \mathcal{E})$ for each $w \in \Omega$ and $U \in \mathcal{C}_2(\mathcal{X}, \mathcal{E})$, and the kernel element $K(\cdot, w)U$ has the reproducing property

$$\langle G, K(\cdot, w)U \rangle_{\mathcal{H}(K) \otimes \mathcal{C}_2(\mathcal{X}, \mathcal{E})} = \langle G(w), U \rangle_{\mathcal{C}_2(\mathcal{X}, \mathcal{E})} := \text{tr}(G(w)U^*).$$

Thus $\mathcal{H}(K) \otimes \mathcal{C}_2(\mathcal{X}, \mathcal{U})$ is a reproducing kernel Hilbert space in the sense of Theorem 2.1 when we identify the range space $\mathcal{L}(\mathcal{E})$ of K as the subspace of $\mathcal{L}(\mathcal{C}_2(\mathcal{X}, \mathcal{E}))$ consisting of left multiplication operators by elements of $\mathcal{L}(\mathcal{E})$:

$$X \in \mathcal{L}(\mathcal{E}) \mapsto L_X \in \mathcal{L}(\mathcal{C}_2(\mathcal{X}, \mathcal{E})) : L_X : T \mapsto XT$$

and we view $\mathcal{C}_2(\mathcal{X}, \mathcal{E})$ as a Hilbert space in the inner product

$$\langle T, T' \rangle_{\mathcal{C}_2(\mathcal{X}, \mathcal{E})} := \text{tr}(TT'^*).$$

In the sequel it will be convenient to use the shorthand notation

$$\mathcal{H}(K)_{\mathcal{X}} := \mathcal{H}(K) \otimes \mathcal{C}_2(\mathcal{X}, \mathcal{E}). \quad (2.8)$$

Note that in this notation, if $\mathcal{H}(K)$ is as in Theorem 2.1, then we have $\mathcal{H}(K) = \mathcal{H}(K)_{\mathbb{C}}$.

Remark 2.3. The space $\mathcal{H}(K)_{\mathcal{X}}$ could just as well have been constructed as equal to the space $\mathcal{H}(K) \otimes \mathcal{C}_2(\mathcal{X}, \mathbb{C})$ where the spaces $\mathcal{H}(K)$ (defined as in Theorem 2.1) and $\mathcal{C}_2(\mathcal{X}, \mathbb{C})$ (the dual version of the Hilbert space \mathcal{X}) are viewed as (\mathbb{C}, \mathbb{C}) -correspondences (i.e., as ordinary Hilbert spaces), and the tensor product reduces to the standard Hilbert-space tensor product.

Suppose that we are given two coefficient Hilbert spaces \mathcal{U} and \mathcal{Y} and an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued function S on Ω . We define the right multiplication operator R_S by

$$(R_S(F))(z) = F(z)S(z).$$

Thus R_S maps $\mathcal{C}_2(\mathcal{U}, \mathcal{E})$ -valued functions on Ω to $\mathcal{C}_2(\mathcal{U}, \mathcal{E})$ -valued functions on Ω . Given a positive $\mathcal{L}(\mathcal{E})$ -valued kernel K on Ω , it is of interest to determine exactly when R_S maps $\mathcal{H}(K)_{\mathcal{Y}}$ boundedly (or contractively) into $\mathcal{H}(K)_{\mathcal{U}}$. The answer is given by the following theorem.

Theorem 2.4. *Let K be an $\mathcal{L}(\mathcal{E})$ -valued positive kernel on Ω and S an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued function on Ω . Then the right multiplication operator R_S is bounded as an operator from $\mathcal{H}(K)_{\mathcal{Y}}$ to $\mathcal{H}(K)_{\mathcal{U}}$ with $\|R_S\| \leq M$ if and only if the \mathbb{C} -valued kernel*

$$k_{X,S,K,M}(z, w) := \text{tr} [X(w)^*(M^2 I_{\mathcal{U}} - S(w)^* S(z))X(z)K(z, w)] \quad (2.9)$$

is a positive kernel on Ω for each choice of function $X: \Omega \rightarrow \mathcal{C}_2(\mathcal{E}, \mathcal{U})$.

Proof. By rescaling it suffices to consider the case $M = 1$ and $\|R_S\| \leq 1$.

The computation

$$\begin{aligned} \langle R_S f, K(\cdot, w)U \rangle_{\mathcal{H}(K)_{\mathcal{U}}} &= \langle f(w)S(w), U \rangle_{\mathcal{C}_2(\mathcal{U}, \mathcal{E})} \\ &= \text{tr}(f(w)S(w)U^*) \\ &= \text{tr}(f(w)(US(w)^*)) \\ &= \langle f, K(\cdot, w)US(w)^* \rangle_{\mathcal{H}(K)_{\mathcal{Y}}} \end{aligned}$$

shows that

$$(R_S)^* : K(\cdot, w)U \mapsto K(\cdot, w)US(w)^*$$

whenever R_S is well defined as an element of $\mathcal{L}(\mathcal{H}(K)_{\mathcal{Y}}, \mathcal{H}(K)_{\mathcal{U}})$. As elements of the form $\sum_{j=1}^N K(\cdot, z_j)U_j$ are dense in $\mathcal{H}(K)_{\mathcal{U}}$, we see that $\|R_S\| \leq 1$ holds if and

only if

$$\begin{aligned} 0 &\leq \left\| \sum_{j=1}^N K(\cdot, z_j) U_j \right\|^2 - \left\| R_S^* \left(\sum_{j=1}^N K(\cdot, z_j) U_j \right) \right\|^2 \\ &= \left\| \sum_{j=1}^N K(\cdot, z_j) U_j \right\|^2 - \left\| \sum_{j=1}^N K(\cdot, z_j) U_j S(z_j)^* \right\|^2 \end{aligned}$$

holds for all choices of $z_1, \dots, z_N \in \Omega$ and $U_1, \dots, U_N \in \mathcal{C}_2(\mathcal{U}, \mathcal{E})$ and $N = 1, 2, \dots$. Expanding out self inner products and using the invariance of the trace under cyclic permutations converts this condition to

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^N \operatorname{tr} (K(z_i, z_j) U_j U_i^* - K(z_i, z_j) U_j S(z_j)^* S(z_i) U_i^*) \\ &= \sum_{i,j=1}^N \operatorname{tr} (U_j (I - S(z_j)^* S(z_i)) U_i^* K(z_i, z_j)) \\ &= \sum_{i,j=1}^N \operatorname{tr} (X(z_j)^* (I - S(z_j)^* S(z_i)) X(z_i) K(z_i, z_j)) \end{aligned}$$

where we have set $X(z_i) = U_i^*$. This positivity condition holding for all choices of $z_1, \dots, z_N \in \Omega$ and $X(z_1), \dots, X(z_N) \in \mathcal{C}_2(\mathcal{E}, \mathcal{U})$ for all $N = 1, 2, \dots$ in turn is equivalent to the positivity of the kernel $k_{X,S,K,1}$ on Ω for all choices of $X: \Omega \rightarrow \mathcal{C}_2(\mathcal{E}, \mathcal{U})$. \square

We shall also need a characterization of functional Hilbert spaces of the form $\mathcal{H}(K)_\mathcal{X}$.

Theorem 2.5. *Suppose that \mathcal{H} is a Hilbert space whose elements are $\mathcal{C}_2(\mathcal{X}, \mathcal{E})$ -valued functions on Ω . Then there is an $\mathcal{L}(\mathcal{E})$ -valued positive kernel K on Ω such that \mathcal{H} is isometrically equal to $\mathcal{H}(K)_\mathcal{X}$ if and only if*

- (1) *the point evaluation map $\mathbf{ev}_w: f \mapsto f(w)$ defines a bounded operator from \mathcal{H} into $\mathcal{C}_2(\mathcal{X}, \mathcal{E})$ for each $w \in \Omega$, and*
- (2) *\mathcal{H} is a right module over $\mathcal{L}(\mathcal{X})$ with the right action of $\mathcal{L}(\mathcal{X})$ commuting with each point evaluation map \mathbf{ev}_w :*

$$\mathbf{ev}_w(f \cdot X) = (\mathbf{ev}_w f)X \text{ or } (f \cdot X)(w) = f(w)X \text{ for all } w \in \Omega. \quad (2.10)$$

Proof. By Theorem 2.1, from the fact that the point evaluations \mathbf{ev}_w are bounded, we get that $\mathcal{H} = \mathcal{H}(\mathbf{K})$ for an $\mathcal{L}(\mathcal{C}_2(\mathcal{X}, \mathcal{E}))$ -valued positive kernel $\mathbf{K}(z, w) = \mathbf{ev}_z \cdot (\mathbf{ev}_w)^*$. The additional condition (2.10) then implies that $\mathbf{K}(z, w)$ commutes with the right multiplication operators $R_X: T \mapsto TX$ on $\mathcal{C}_2(\mathcal{X}, \mathcal{E})$ ($X \in \mathcal{L}(\mathcal{X})$). This is enough to force $\mathbf{K}(z, w)$ to be a left multiplication operator $\mathbf{K}(z, w) = L_{K(z,w)}$ for a $K(z, w) \in \mathcal{L}(\mathcal{E})$. One next verifies that K so constructed is an $\mathcal{L}(\mathcal{E})$ -valued positive kernel and that we recover \mathcal{H} as $\mathcal{H} = \mathcal{H}(K)_\mathcal{X}$. \square

We shall also have use for a far-reaching generalization of the positive kernels discussed so far introduced by Barreto, Bhat, Liebscher, and Skeide in [24]. Given

two C^* -algebras \mathfrak{A} and \mathfrak{B} , we say that a function Γ on $\Omega \times \Omega$ with values in $\mathcal{L}(\mathfrak{A}, \mathfrak{B})$ is a *completely positive kernel* if

$$\sum_{i,j=1}^N b_i^* \Gamma(z_i, z_j) [a_i^* a_j] b_j \geq 0 \text{ (in } \mathfrak{B}) \quad (2.11)$$

for all choices of $z_1, \dots, z_N \in \Omega$, $a_1, \dots, a_N \in \mathfrak{A}$, $b_1, \dots, b_N \in \mathfrak{B}$ for all $N = 1, 2, \dots$. The following characterization of completely positive kernels is the completely positive parallel to Theorems 2.1 and 2.2.

Theorem 2.6. (See [24, 15].) *Given a function Γ on $\Omega \times \Omega$ with values in $\mathcal{L}(\mathfrak{A}, \mathfrak{B})$, the following are equivalent:*

- (1) Γ is a completely positive kernel, i.e., condition (2.11) holds.
- (2) There is an $(\mathfrak{A}, \mathfrak{B})$ -correspondence $\mathcal{H}(\Gamma)$ whose elements consist of \mathfrak{B} -valued functions f on Ω such that $K(\cdot, w)[a] \in \mathcal{H}(\Gamma)$ for each $w \in \Omega$ and $a \in \mathfrak{A}$ and such that

$$\langle f, K(\cdot, w)[a] \rangle_{\mathcal{H}(\Gamma)} = (a^* \cdot f)(w)$$

for all $f \in \mathcal{H}(\Gamma)$, $a \in \mathfrak{A}$, and $w \in \Omega$.

- (3) K has a Kolmogorov decomposition of the following form: there is an $(\mathfrak{A}, \mathfrak{B})$ -correspondence \mathcal{H} and a function H on Ω with values in the space $\mathcal{L}(\mathcal{H}, \mathfrak{B})$ of adjointable operators from \mathcal{H} to \mathfrak{B} so that

$$K(z, w)[a] = H(z)\pi(a)H(w)^*.$$

Here $a \mapsto \pi(a)$ represents the left \mathfrak{A} -action on \mathcal{H} : $\pi(a)f = a \cdot f$ for $f \in \mathcal{H}$.

In case $\mathfrak{B} = \mathcal{L}(\mathcal{E})$ for a Hilbert space \mathcal{E} , then we also have Hilbert space versions of conditions (2) and (3):

- (2') There is an $(\mathfrak{A}, \mathbb{C})$ -correspondence $\mathcal{H}(\Gamma)$ (i.e., a Hilbert space $\mathcal{H}(\Gamma)$ equipped with a $*$ -representation $\pi: \mathfrak{A} \rightarrow \mathcal{L}(\mathcal{H}(\Gamma))$ of \mathfrak{A}) whose elements are \mathcal{E} -valued functions f on Ω such that $K(\cdot, w)[a]e \in \mathcal{H}(\Gamma)$ for each $w \in \Omega$, $a \in \mathfrak{A}$, $e \in \mathcal{E}$, and such that

$$\langle f, K(\cdot, w)[a]e \rangle_{\mathcal{H}(\Gamma)} = \langle (a^* \cdot f)(w), e \rangle_{\mathcal{E}}$$

for all $f \in \mathcal{H}(\Gamma)$, $a \in \mathfrak{A}$, $w \in \Omega$.

- (3') There exists a Hilbert space \mathcal{H} carrying a $*$ -representation π of \mathfrak{A} and there exists a function $H: \Omega \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{E})$ so that

$$K(z, w)[a] = H(z)\pi(a)H(w)^*.$$

Remark 2.7. The positivity condition in Theorem 2.4 can be equivalently formulated as the condition that the kernel

$$k_{\Gamma, S, K}(z, w) = [\Gamma(z, w)[I - S(w)^* S(z)], K(z, w)]_{\mathcal{C}_1(\mathcal{E}) \times \mathcal{L}(\mathcal{E})}$$

be a positive \mathbb{C} -valued kernel on Ω for every choice of completely positive kernel

$$\Gamma: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{L}(\mathcal{U}), \mathcal{C}_1(\mathcal{E})),$$

where the outside bracket

$$[\cdot, \cdot]_{\mathcal{C}_1(\mathcal{E}) \times \mathcal{L}(\mathcal{E})}$$

is the duality pairing between the trace-class operators $\mathcal{C}_1(\mathcal{E})$ and the bounded linear operators $\mathcal{L}(\mathcal{E})$.

2.3. Ψ -unitary colligations. For the transfer-function realization

$$S(z) = D + zC(I - zA)^{-1}B$$

in the operator-valued test-function setting to be developed in the sequel, we shall need a more elaborate version of the unitary colligation matrix $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ which we now describe. Given a collection of test functions Ψ as in Section 2.1, as described there we view Ψ as a completely regular topological space. Then the space $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$ of bounded $\mathcal{L}(\mathcal{Y}_T)$ -valued functions on Ψ is a C^* -algebra while the space $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T))$ of continuous $\mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)$ -valued functions is not (unless $\mathcal{U}_T = \mathcal{Y}_T$). However we may view $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T))$ as a $(C_b(\Psi, \mathcal{U}_T), C_b(\Psi, \mathcal{Y}_T))$ -correspondence, with $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$ -valued inner product given by

$$(\langle F, F' \rangle_{C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T))}) (\psi) := F'(\psi)^* F(\psi).$$

If \mathcal{X} is a Hilbert space carrying a $*$ -representation ρ of $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$, then we may view \mathcal{X} as a $(C_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathbb{C})$ correspondence (with the representation ρ providing the left $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$ -action on \mathcal{X}) and form the internal tensor product $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes_\rho \mathcal{X}$. We shall say that a 2×2 -block unitary matrix $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a Ψ -unitary colligation if \mathbf{U} has the form

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes_\rho \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

for \mathcal{X} equal to a Hilbert space carrying a $*$ -representation ρ of $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$.

A particular element of $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T))$ is the function $\mathbb{E}(z)^*$, where $\mathbb{E}(z)$ is as in (2.2) (for a given $z \in \Omega$). Hence the tensor multiplication operator

$$L_{\mathbb{E}(z)^*} : x \mapsto \mathbb{E}(z)^* \otimes x \quad (2.12)$$

defines an operator from \mathcal{X} to $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes_\rho \mathcal{X}$; one can verify that its adjoint acting on pure tensors is given by

$$L_{\mathbb{E}(z)^*}^* : g \otimes x \mapsto \rho(\mathbb{E}(z)g)x.$$

As a consequence we get the identity

$$L_{\mathbb{E}(z)^*}^* L_{\mathbb{E}(w)^*} x = L_{\mathbb{E}(z)^*}^* (\mathbb{E}(w)^* \otimes x) = \rho(\mathbb{E}(z)\mathbb{E}(w)^*) x. \quad (2.13)$$

In case $\mathcal{Y}_T = \mathcal{U}_T$ (the square case), then $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes_\rho \mathcal{X}$ collapses down to \mathcal{X} (a consequence of the balancing property (2.7)), and then $L_{\mathbb{E}(z)^*}^*$ can be identified with $L_{\mathbb{E}(z)^*}^* = \rho(\mathbb{E}(z))$. We conclude that the tensor-product construction is exactly the technical tool needed to push the square case to the non-square case. This type of colligation matrix appears in [8, 27, 29] for the square case and in [44] for the nonsquare case.

3. THE SCHUR-AGLER CLASS ASSOCIATED WITH A COLLECTION OF TEST FUNCTIONS

Suppose that we are given a collection Ψ of test functions $\psi: \Omega \rightarrow \mathcal{L}(\mathcal{U}_T, \mathcal{Y}_T)$ satisfying the admissibility condition (2.1). For \mathcal{E} any auxiliary Hilbert space and K an $\mathcal{L}(\mathcal{E})$ -valued positive kernel on Ω , we say that K is Ψ -admissible, written as $K \in \mathcal{K}_\Psi(\mathcal{E})$, if the operator $R_\psi: f(z) \mapsto f(z)\psi(z)$ is contractive from $\mathcal{H}(K)_{\mathcal{Y}_T}$ to $\mathcal{H}(K)_{\mathcal{U}_T}$ for each $\psi \in \Psi$, or equivalently (by Theorem 2.4), if the \mathbb{C} -valued kernel

$$k_{X, \psi, K}(z, w) = \text{tr}(X(w)^*(I - \psi(w)^*\psi(z))X(z)K(z, w)) \quad (3.1)$$

is a positive kernel for each choice of $X: \Omega \rightarrow \mathcal{C}_2(\mathcal{E}, \mathcal{U}_T)$ and $\psi \in \Psi$. We then say that the function $S: \Omega \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is in the Ψ -Schur-Agler class $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$ if the operator R_S of right multiplication by S is contractive from $\mathcal{H}(\mathcal{Y})_{\mathcal{Y}}$ to $\mathcal{H}(\mathcal{Y})_{\mathcal{U}}$ for each Ψ -admissible $\mathcal{L}(\mathcal{Y})$ -valued positive kernel K , or equivalently, if the kernel

$$k_{Y,S,K}(z, w) = \text{tr}(Y(w)^*(I - S(w)^*S(z))Y(z)K(z, w)) \quad (3.2)$$

is a positive \mathbb{C} -valued kernel for each choice of $Y: \Omega \rightarrow \mathcal{C}_2(\mathcal{Y}, \mathcal{U})$ and $K \in \mathcal{K}_\Psi(\mathcal{Y})$.

Our main result on the Schur-Agler class $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$ is the following.

Theorem 3.1. *Suppose that we are given a collection of test functions Ψ satisfying condition (2.1) and S_0 is a function on some subset Ω_0 of Ω with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. Consider the following conditions:*

- (1) S_0 can be extended to a function S defined on all of Ω such that $S \in \mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$, i.e., the kernel (3.2) is a positive kernel for all choices of $\mathcal{L}(\mathcal{Y}, \mathcal{U})$ -valued functions Y on Ω_0 and all choices of kernels $K \in \mathcal{K}_\Psi(\mathcal{Y})$.
- (2) S_0 has an Agler decomposition on Ω_0 , i.e., there is a completely positive kernel $\Gamma: \Omega_0 \times \Omega_0 \rightarrow \mathcal{L}(C_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathcal{L}(\mathcal{Y}))$ so that

$$I - S_0(z)S_0(w)^* = \Gamma(z, w)[I - \mathbb{E}(z)\mathbb{E}(w)^*] \quad (3.3)$$

for all $z, w \in \Omega_0$ (where $\mathbb{E}(z) \in C_b(\Psi, \mathcal{L}(\mathcal{U}_T, \mathcal{Y}_T))$ is as in (2.2)).

- (3) There is a Hilbert state space \mathcal{X} which carries a $*$ -representation of the C^* -algebra $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$ and a Ψ -unitary colligation \mathbf{U} (see Section 2.3)

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes_\rho \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \quad (3.4)$$

so that S_0 has the transfer-function realization

$$S_0(z) = D + C(I - L_{\mathbb{E}(z)^*}^* A)^{-1} L_{\mathbb{E}(z)^*}^* B \quad (3.5)$$

for $z \in \Omega_0$.

Then (1) \Rightarrow (2) \Leftrightarrow (3); if $\dim \mathcal{Y}_T < \infty$, then also (2) \Rightarrow (1) and hence (1), (2), (3) are all equivalent to each other.

We shall prove (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (2) and, if $\dim \mathcal{Y}_T < \infty$, then also (2) \Rightarrow (1).

Proof of (1) \Rightarrow (2): Step 1: Ω_0 is a finite subset of Ω .

We define a cone \mathcal{C} by

$$\mathcal{C} = \{\Xi: \Omega_0 \times \Omega_0 \rightarrow \mathcal{L}(\mathcal{Y}): \Xi(z, w) = \Gamma(z, w)[I - \mathbb{E}(z)\mathbb{E}(w)^*] \text{ for some completely positive kernel } \Gamma: \Omega_0 \times \Omega_0 \rightarrow \mathcal{L}(C_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathcal{L}(\mathcal{Y}))\}.$$

Note that the elements of \mathcal{C} can be viewed as matrices with rows and columns indexed by the finite set Ω_0 and matrix entries in $\mathcal{L}(\mathcal{Y})$. Thus we may view \mathcal{C} as a subset of the linear space \mathcal{V} of all such matrices with topology of pointwise weak-* convergence. We shall need a few preliminary lemmas. It is easy to verify that \mathcal{C} is a cone in \mathcal{V} .

Lemma 3.2. *The cone \mathcal{C} is closed in \mathcal{V} .*

Proof of Lemma. Suppose that $\{\Xi_\alpha\}$ is a net of elements of \mathcal{C} such that $\{\Xi_\alpha(z, w)\}$ converges weak-* to $\Xi(z, w)$ for each $z, w \in \Omega_0$. Thus, for each index α there is a choice of completely positive kernel Γ_α so that

$$\Xi_\alpha(z, w) = \Gamma_\alpha(z, w)[I - \mathbb{E}(z)\mathbb{E}(w)^*]. \quad (3.6)$$

The computation

$$\begin{aligned}
\Gamma_\alpha(z, z)[I] &= \Gamma_\alpha(z, z)[(I - \mathbb{E}(z)\mathbb{E}(z)^*)^{1/2}(I - \mathbb{E}(z)\mathbb{E}(w)^*)^{-1}(I - \mathbb{E}(z)\mathbb{E}(z)^*)^{1/2}] \\
&\leq \Gamma_\alpha(z, z) \left[(I - \mathbb{E}(z)\mathbb{E}(z)^*)^{1/2} \left(\frac{1}{1 - \|\mathbb{E}(z)\|^2} \right) (I - \mathbb{E}(z)\mathbb{E}(z)^*)^{1/2} \right] \\
&= \left(\frac{1}{1 - \|\mathbb{E}(z)\|^2} \right) \Gamma_\alpha(z, z)[I - \mathbb{E}(z)\mathbb{E}(z)^*] \\
&= \left(\frac{1}{1 - \|\mathbb{E}(z)\|^2} \right) \Xi_\alpha(z, z)
\end{aligned}$$

shows that

$$\|\Gamma_\alpha(z, z)\| \leq M_z \|\Xi_\alpha(z, z)\| \text{ where } M_z = \frac{1}{1 - \|\mathbb{E}(z)\|^2}, \quad (3.7)$$

where we used here the underlying assumption (2.1) for our set of test functions Ψ . Since the block 2×2 matrix

$$\begin{bmatrix} \Gamma_\alpha(z, z)[I] & \Gamma_\alpha(z, w)[I] \\ \Gamma_\alpha(w, z)[I] & \Gamma_\alpha(w, w)[I] \end{bmatrix}$$

is positive semidefinite for each index α and each pair of points $z, w \in \Omega_0$, it follows that

$$\|\Gamma_\alpha(z, w)\| \leq M_z M_w \|\Xi_\alpha(z, w)\|^{1/2} \|\Xi_\alpha(w, w)\|^{1/2}. \quad (3.8)$$

Since Ω_0 is finite, we see that $\|\Gamma_\alpha(z, w)\|$ is in fact bounded uniformly with respect to the indices α and the points z, w in Ω_0 . Since $\mathcal{L}(C_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathcal{L}(\mathcal{Y}))$ is the Banach-space dual of the projective tensor-product Banach space $\mathcal{C}_1(\mathcal{Y}) \otimes C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$ (see e.g. [52, Theorem IV.2.3]), it follows from the Banach-Alaoglu theorem that there is a subnet $\{\Gamma_\beta\}$ of $\{\Gamma_\alpha\}$ such that $\{\Gamma_\beta(z, w)\}$ converges weak-* to some $\Gamma_\infty(z, w) \in \mathcal{L}(C_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathcal{L}(\mathcal{Y}))$. It is straightforward to verify that the defining property (2.11) for a completely positive kernel is preserved under such weak-* limits; hence Γ_∞ is again a completely positive kernel. Moreover, from the fact that $\{\Xi_\alpha(z, w)\}$ converges weak-* to $\Xi(z, w)$, we get that the subnet $\{\Xi_\beta(z, w)\}$ also converges weak-* to $\Xi(z, w)$. Taking limits in the formula (3.6) leads us to the representation

$$\Xi(z, w) = \Gamma_\infty(z, w)[I - \mathbb{E}(z)\mathbb{E}(w)^*]$$

for the limit kernel $\Xi(z, w)$. We conclude that the limit kernel Ξ is again in \mathcal{C} as wanted. \square

Lemma 3.3. *Suppose that $\Xi(z, w) = H(z)H(w)^*$ is a positive $\mathcal{L}(\mathcal{Y})$ -valued kernel on Ω_0 . Then Ξ is in \mathcal{C} .*

Proof of Lemma. Let us say that $\Xi(z, w) = H(z)H(w)^*$ where $H: \Omega \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$ for some coefficient Hilbert space \mathcal{X} . Let ψ_0 be any fixed test function in Ψ . It suffices to find another coefficient Hilbert space $\tilde{\mathcal{X}}$ and a function $G: \Omega_0 \rightarrow \mathcal{L}(\tilde{\mathcal{X}} \otimes \mathcal{Y}_T, \mathcal{Y})$ so that

$$\Xi(z, w) = G(z) (I_{\tilde{\mathcal{X}}} \otimes (I - \psi_0(z)\psi_0(w)^*)) G(w)^*,$$

for then we have the needed representation $\Xi(z, w) = \Gamma_0(z, w)[I - \mathbb{E}(z)\mathbb{E}(w)^*]$ with Γ_0 given by

$$\Gamma_0(z, w)[g] = G(z)(I_{\tilde{\mathcal{X}}} \otimes g(\psi_0))G(w)^*.$$

Toward this end, choose a unit vector y_0 in \mathcal{Y}_T and note that

$$y_0^*(I - \psi_0(z)\psi_0(w)^*)y_0 = 1 - y_0^*\psi_0(z)\psi_0(w)^*y_0$$

is invertible (as an element of \mathbb{C}) by our underlying assumption (2.1). Moreover we have the geometric series representation for the inverse:

$$\frac{1}{1 - y_0^* \psi_0(z) \psi_0(w)^* y_0} = \sum_{n=0}^{\infty} (y_0^* \psi_0(z) \psi_0(w)^* y_0)^n \quad (3.9)$$

where each term $(y_0^* \psi_0(z) \psi_0(w)^* y_0)^n$ is a positive kernel due to the Schur multiplier theorem (see e.g. [48, Theorem 3.7]). Thus there exist functions $g_n: \Omega_0 \rightarrow \mathcal{L}(\tilde{\mathcal{G}}_n, \mathbb{C})$ so that

$$(y_0^* \psi_0(z) \psi_0(w)^* y_0)^n = g_n(z) g_n(w)^*.$$

Then we may rewrite (3.9) as

$$\frac{1}{1 - y_0^* \psi_0(z) \psi_0(w)^* y_0} = \sum_{n=0}^{\infty} g_n(z) g_n(w)^*. \quad (3.10)$$

We conclude that

$$\begin{aligned} \Xi(z, w) &= H(z) H(w)^* \\ &= H(z) \left(\frac{1}{1 - y_0^* \psi_0(z) \psi_0(w)^* y_0} \cdot (1 - y_0^* \psi_0(z) \psi_0(w)^* y_0) I_{\mathcal{X}} \right) H(w)^* \\ &= \sum_{n=0}^{\infty} H(z) (g_n(z) g_n(w)^* (1 - y_0^* \psi_0(z) \psi_0(w)^* y_0) I_{\mathcal{X}}) H(w)^* \\ &= \sum_{n=0}^{\infty} H(z) g_n(z) \left((1 - y_0^* \psi_0(z) \psi_0(w)^* y_0) I_{\tilde{\mathcal{G}}_n} \right) g_n(w)^* H(w)^* \\ &= \sum_{n=0}^{\infty} H(z) (g_n(z) \otimes y_0^*) \left(I_{\tilde{\mathcal{G}}_n} \otimes (I - \psi_0(z) \psi_0(w)^*) \right) (g_n(w)^* \otimes y_0) H(w)^* \\ &= G(z) (I_{\tilde{\mathcal{X}}} \otimes (I - \psi_0(z) \psi_0(w)^*)) G(w)^* \end{aligned}$$

where we set

$$G(z) = \begin{bmatrix} H(z)(g_1(z) \otimes y_0^*) & H(z)(g_2(z) \otimes y_0^*) & \cdots \end{bmatrix}, \quad \tilde{\mathcal{X}} = \bigoplus_{n=1}^{\infty} \tilde{\mathcal{G}}_n.$$

□

Let us now note that the assertion of the condition (2) in the statement of the Theorem is that the kernel $\Xi_{S_0}(z, w) := I - S_0(z) S_0(w)^*$ is in \mathcal{C} . As \mathcal{V} is a locally convex linear topological vector space and \mathcal{C} is closed in \mathcal{V} , by a standard Hahn-Banach separation principle (see [51, Theorem 3.49b]), to show that $\Xi_S \in \mathcal{C}$ it suffices to show: $\operatorname{Re} \mathbb{L}(\Xi_S) \geq 0$ whenever \mathbb{L} is a continuous linear functional on \mathcal{V} such that $\operatorname{Re} \mathbb{L}(\Xi) \geq 0$ for each $\Xi \in \mathcal{C}$.

With this strategy in mind let us suppose that \mathbb{L} is a continuous linear functional on \mathcal{V} such that $\operatorname{Re} \mathbb{L}(\Xi) \geq 0$ for each $\Xi \in \mathcal{C}$. We then define \mathbb{L}_1 on \mathcal{V} by

$$\mathbb{L}_1(\Xi) = \frac{1}{2} \left(\mathbb{L}(\Xi) + \overline{\mathbb{L}(\Xi^\vee)} \right)$$

where we set

$$\Xi^\vee(z, w) = \Xi(w, z)^*.$$

Easy properties are that

$$\mathbb{L}_1(\Xi) = \operatorname{Re} \mathbb{L}(\Xi) \text{ if } \Xi^\vee = \Xi. \quad (3.11)$$

For $\epsilon > 0$ be an arbitrarily small but positive number, we use the functional \mathbb{L}_1 to define an inner product on the space $\mathcal{H}_{\mathbb{L}_1, \epsilon}$ of functions $f: \Omega_0 \rightarrow \mathcal{Y}$ by

$$\langle f, g \rangle_{\mathcal{H}_{\mathbb{L}_1, \epsilon}} = \mathbb{L}_1(\Delta_{f,g}) + \epsilon^2 \sum_{w \in \Omega_0} \text{tr}(\Delta_{f,g}(w, w))$$

where we have set

$$\Delta_{f,g}(z, w) = f(z)g(w)^*. \quad (3.12)$$

By Lemma 3.3 we know that $\Delta_{f,f} \in \mathcal{C}$ and hence $\text{Re } \mathbb{L}(\Delta_{f,f}) \geq 0$. Since $\Delta_{f,f} = \Delta_{f,f}^\vee$, as a consequence of (3.11) we know that $\text{Re } \mathbb{L}(\Delta_{f,f}) = \mathbb{L}_1(\Delta_{f,f})$. From these observations it follows that $\langle \cdot, \cdot \rangle_{\mathcal{H}_{\mathbb{L}_1, \epsilon}}$ is a positive semidefinite inner product. Hence we can take the Hausdorff completion of $\mathcal{H}_{\mathbb{L}_1, \epsilon}$ to arrive at a Hilbert space, still denoted as $\mathcal{H}_{\mathbb{L}_1, \epsilon}$.

For \mathcal{X} a coefficient Hilbert space, we shall be interested in the space $\mathcal{H}_{\mathbb{L}_1, \epsilon} \otimes \mathcal{C}_2(\mathcal{X}, \mathbb{C})$. The following lemma is crucial.

Lemma 3.4. *The space $\mathcal{H}_{\mathbb{L}_1, \epsilon} \otimes \mathcal{C}_2(\mathcal{X}, \mathbb{C})$ can be identified with the space $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{X}}$ consisting of $\mathcal{C}_2(\mathcal{X}, \mathcal{Y})$ -valued functions f on Ω with inner product given by*

$$\langle f, g \rangle_{(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{X}}} = \mathbb{L}_1(\Delta_{f,g}) + \epsilon^2 \sum_{w \in \Omega_0} \text{tr}(\Delta_{f,g}(w, w)) \quad (3.13)$$

where $\Delta_{f,g}$ has the same form as in (3.12) (but where now the middle space is \mathcal{X} rather than \mathbb{C}):

$$\Delta_{f,g}(z, w) = f(z)g(w)^*.$$

Proof of lemma. For convenience of notation we drop the ϵ -term in the inner product as the $\epsilon > 0$ case proceeds in the same way but with more cumbersome notation. For $f \otimes x^*$ a pure tensor in $\mathcal{H}_{\mathbb{L}_1} \otimes \mathcal{C}_2(\mathcal{X}, \mathbb{C})$ (so $f \in \mathcal{H}_{\mathbb{L}_1}$ and $x \in \mathcal{X} \cong \mathcal{L}(\mathbb{C}, \mathcal{X})$) and similarly for $f' \otimes x'^*$, we have

$$\begin{aligned} \langle f \otimes x^*, f' \otimes x'^* \rangle_{\mathcal{H}_{\mathbb{L}_1} \otimes \mathcal{C}_2(\mathcal{X}, \mathbb{C})} &= \langle \langle f, f' \rangle_{\mathcal{H}_{\mathbb{L}_1}} x^*, x'^* \rangle_{\mathcal{C}_2(\mathcal{X}, \mathbb{C})} \\ &= \mathbb{L}_1(\Delta_{f,f'}) x^* x'^* = \mathbb{L}_1(\Delta_{f,f'} x^* x'^*) \end{aligned}$$

where the last step follows since $x^* x'$ is just a complex number. Next observe that

$$\begin{aligned} \Delta_{f,f'}(z, w) x^* x' &= f(z) f'(w)^* (x^* x') = f(z) (x^* x') f'(w)^* \\ &= (f(z) x^*) (f'(w) x'^*)^* = \Delta_{f, x^*, f', x'^*}(z, w). \end{aligned}$$

By extending this calculation to linear combinations of pure tensors, the result follows. \square

With the formulation of the space $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{X}}$ in hand, it makes sense to ask whether the right multiplication operator $R_\psi: f(z) \mapsto f(z)\psi(z)$ defines a contraction operator from $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}_T}$ to $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{U}_T}$. The answer is given by the next lemma.

Lemma 3.5. *For each test function $\psi \in \Psi$, the right multiplication operator R_ψ defines a contraction operator from $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}_T}$ to $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{U}_T}$.*

Proof of Lemma. R_ψ is contractive if and only if

$$\|f\|_{(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}_T}}^2 - \|R_\psi f\|_{(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{U}_T}}^2 \geq 0$$

for all $f \in (\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}_T}$. This translates to the condition that

$$\mathbb{L}_1(\Delta_{f,f} - \Delta_{f\psi, f\psi}) + \epsilon^2 \sum_{w \in \Omega_0} [\Delta_{f,f}(w, w) - \Delta_{f\psi, f\psi}(w, w)] \geq 0$$

for all such f . Observe that

$$\Delta_{f,f}(z, w) - \Delta_{f\psi, f\psi}(z, w) = f(z)(I - \psi(z)\psi(w)^*)f(w)^*$$

from which we see that the kernel $\Xi := \Delta_{f,f} - \Delta_{f\psi, f\psi}$ is in the cone \mathcal{C} : note that the kernel $\Gamma(z, w)[g] = f(z)g(\psi)f(w)^*$ is completely positive since its Kolmogorov decomposition (condition (3') in Theorem 2.6) is exhibited. Thus $\text{Re } \mathbb{L}(\Xi) \geq 0$, and hence, since $\Xi = \Xi^\vee$, also $\mathbb{L}_1(\Xi) \geq 0$. The ϵ -term is also nonnegative since $\|\psi(w)\| < 1$ for each $w \in \Omega_0$. It now follows that $\|R_\psi\| \leq 1$ as asserted. \square

To make use of the hypothesis that $S \in \mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$, we need to convert the space $\mathcal{H}_{\mathbb{L}_1, \epsilon}$ to a reproducing kernel space. This is done as follows; it is at this point that we make use of the ϵ -regularization of the $\mathcal{H}_{\mathbb{L}_1}$ -inner product.

Lemma 3.6. *The space $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}}$ is isometrically equal to a reproducing kernel Hilbert spaces $\mathcal{H}(K)_{\mathcal{Y}}$ for a positive kernel $K \in \mathcal{K}_\Psi(\mathcal{Y})$.*

Proof of lemma. We wish to apply Theorem 2.5 with \mathcal{E} and \mathcal{X} equal to \mathcal{Y} and with Ω_0 equal to Ω . To this end, we note that elements of $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}}$ are $\mathcal{C}_2(\mathcal{Y})$ -valued functions, at least on the dense set before the Hausdorff-completion step is carried out in the construction of the space. However, the presence of the term with the ϵ^2 factor in the definition of the $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}}$ -inner product guarantees that the point-evaluation map $\mathbf{ev}_w: (\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}} \rightarrow \mathcal{C}_2(\mathcal{Y})$ is bounded with norm at most $1/\epsilon$. Hence condition (1) in Theorem 2.5 is verified. Condition (2) is straightforward since $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}}$ is itself a tensor-product space $\mathcal{H}_{\mathbb{L}_1, \epsilon} \otimes \mathcal{C}_2(\mathcal{Y}, \mathbb{C})$. We conclude that $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}}$ is isometrically equal to a reproducing kernel Hilbert space $\mathcal{H}(K)_{\mathcal{Y}}$ for a uniquely determined $\mathcal{L}(\mathcal{Y})$ -valued positive kernel K .

Finally we must verify that K is Ψ -admissible. But this is an immediate consequence of Lemma 3.5. \square

To conclude the proof of Step 1 (the case where Ω_0 is finite), we proceed as follows. Let K be the positive kernel identified in Lemma 3.6. Since $K \in \mathcal{K}_\Psi(\mathcal{Y})$, we use the assumption that S is in the Schur-Agler class $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$ to conclude that the operator R_S of right multiplication by S is contractive from $\mathcal{H}(K)_{\mathcal{Y}}$ to $\mathcal{H}(K)_{\mathcal{U}}$. As Lemma 3.6 also tells us that $\mathcal{H}(K)_{\mathcal{Y}}$ is isometrically equal to $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}}$, trivially we can also say that R_S is contractive from $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}}$ to $(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{U}}$. The criterion for this to be the case is that

$$\|f\|_{(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}}}^2 - \|R_S f\|_{(\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{U}}}^2 \geq 0 \text{ for all } f \in (\mathcal{H}_{\mathbb{L}_1, \epsilon})_{\mathcal{Y}},$$

or, equivalently

$$\mathbb{L}_1(\Delta_{f,f} - \Delta_{fS_0, fS_0}) + \epsilon^2 \sum_{w \in \Omega_0} \text{tr}(\Delta_{f,f}(w, w) - \Delta_{fS_0, fS_0}(w, w)) \geq 0 \text{ for all } f,$$

where $\Delta_{f,f}(z, w) - \Delta_{fS_0, fS_0}(z, w) = f(z)\Xi_S(z, w)f(w)^*$. In particular, taking $f(z) = P_n$ for all $z \in \Omega_0$ where $\{P_n\}$ is an increasing sequence of finite-rank orthogonal projections converging strongly to the identity operator $I_{\mathcal{Y}}$ gives us

$$\mathbb{L}_1(P_n \Xi_{S_0} P_n) + \epsilon^2 \sum_{z \in \Omega_0} \text{tr}(P_n \Xi_{S_0}(z, z) P_n) \geq 0.$$

As this holds for all $\epsilon > 0$, we may take the limit as $\epsilon \rightarrow 0$ (while holding n fixed) to get

$$\mathbb{L}_1(P_n \Xi_{S_0} P_n) \geq 0 \quad (3.14)$$

for all n . By the weak-* continuity of \mathbb{L}_1 we have that

$$\lim_{n \rightarrow \infty} \mathbb{L}_1(P_n \Xi_{S_0} P_n) = \mathbb{L}_1(\Xi_{S_0}).$$

Taking limits in (3.14) then gives us $\mathbb{L}_1(\Xi_{S_0}) \geq 0$. As $\Xi_{S_0} = \Xi_{S_0}^\vee$, this gives us finally $\operatorname{Re} \mathbb{L}(\Xi_{S_0}) \geq 0$ as required, and we conclude that $S_0 \in \mathcal{C}$ as wanted. This concludes the proof of Step 1.

Step 2: Ω_0 is not necessarily finite.

We now remove that assumption that Ω_0 is finite. It is now understood how this step is efficiently handled as an application of the Kurosh Theorem (see [27, 29]). By Step 1, we know that for each finite subset Ω_F of Ω , there is an associated completely positive kernel Γ_F (not necessarily uniquely determined) so that the Agler decomposition

$$\Xi_{S_0}(z, w) := I - S_0(z)S_0(w)^* = \Gamma_{\Omega_F}(z, w)[I - \mathbb{E}(z)\mathbb{E}(w)^*] \quad (3.15)$$

holds for all $z, w \in \Omega_F$. To set up the Kurosh Theorem, for each finite subset $\Omega_F \subset \Omega$, we let Φ_{Ω_F} denote the collection

$$\Phi_{\Omega_F} = \{\Xi : \Xi \text{ completely positive kernel such that (3.15) holds for } z, w \in \Omega_F\}.$$

By applying the argument used in the proof of Lemma 3.2, one can see that Φ_{Ω_F} is compact in the pointwise weak-* convergence topology inherited from the space of $\mathcal{L}(C_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathcal{L}(\mathcal{Y}))$ -valued functions on $\Omega \times \Omega$. The Kurosh Theorem (see [11, page 75]) tells us that, for each finite subset Ω_F of Ω , there is a choice of completely positive kernel Γ_{Ω_F} for which (3.15) holds on Ω_F such that, in addition, whenever $\Omega_F, \Omega_{F'}$ are two subsets of Ω with $\Omega_F \subset \Omega_{F'}$, then $\Gamma_{\Omega_{F'}}|_{\Omega_F \times \Omega_F} = \Gamma_{\Omega_F}$. We may then define a completely positive kernel Γ on all of $\Omega \times \Omega$ by

$$\Gamma(z, w) = \Gamma_{\Omega_F}(z, w) \text{ where } \Omega_F \text{ finite, } z, w \in \Omega_F.$$

The construction guarantees that Γ is well defined and the fact that each Γ_{Ω_F} is completely positive on Ω_F guarantees that Γ is completely positive as a kernel on all of Ω . We have now completed the proof of (1) \Rightarrow (2) in Theorem 3.1. \square

Proof of (2) \Rightarrow (3). We are given a completely positive kernel Γ on Ω_0 so that (3.3) holds for $z, w \in \Omega_0$. By condition (3') in Theorem 2.6, Γ has a decomposition of the form

$$\Gamma(z, w)[g] = H(z)\rho(g)H(w)^*$$

where $H: \Omega_0 \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$ for an auxiliary Hilbert space \mathcal{X} which also carries a *-representation ρ of the C^* -algebra $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$. From (3.3) we then deduce

$$\begin{aligned} I - S_0(z)S_0(w)^* &= \Gamma(z, w)[I - \mathbb{E}(z)\mathbb{E}(w)^*] \\ &= H(z)\rho(I - \mathbb{E}(z)\mathbb{E}(w)^*)H(w)^* \\ &= H(z)H(w)^* - H(z)L_{\mathbb{E}(z)^*}^*L_{\mathbb{E}(w)^*}H(w)^* \end{aligned}$$

where we use (2.13). This in turn can be rearranged as

$$H(z)L_{\mathbb{E}(z)^*}^*L_{\mathbb{E}(w)^*}H(w)^* + I = H(z)H(w)^* + S_0(z)S_0(w)^*$$

which leads to the inner product identity

$$\begin{aligned} & \langle L_{\mathbb{E}(w)^*} H(w)^* y_w, L_{\mathbb{E}(z)^*} H(z)^* y_z \rangle_{C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X}} + \langle y_w, y_z \rangle_{\mathcal{Y}} \\ &= \langle H(w)^* y_w, H(z)^* y_z x \rangle + \langle S_0(w)^* y_w, S_0(z)^* y_z \rangle_{\mathcal{U}} \end{aligned}$$

for arbitrary y_w and y_z in \mathcal{Y} . It then follows that the mapping V given by

$$V: \begin{bmatrix} L_{\mathbb{E}(w)^*} H(w)^* y_w \\ y_w \end{bmatrix} \mapsto \begin{bmatrix} H(w)^* y_w \\ S_0(w)^* y_w \end{bmatrix} \quad (3.16)$$

extends by linearity and continuity to a well-defined isometry from the subspace

$$\mathcal{D} := \overline{\text{span}} \left\{ \begin{bmatrix} L_{\mathbb{E}(w)^*} H(w)^* y_w \\ y_w \end{bmatrix} : y_w \in \mathcal{Y}, w \in \Omega \right\} \subset \begin{bmatrix} C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

onto the subspace

$$\mathcal{R} := \overline{\text{span}} \left\{ \begin{bmatrix} H(w)^* y_w \\ S_0(w)^* y_w \end{bmatrix} : y_w \in \mathcal{Y}, w \in \Omega \right\} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}.$$

By replacing \mathcal{X} with $\mathcal{X}' = \mathcal{X} \oplus \tilde{\mathcal{X}}$ where $\tilde{\mathcal{X}}$ is an infinite-dimensional Hilbert space if necessary, we can arrange that the defect spaces $\begin{bmatrix} \mathcal{X}' \\ \mathcal{Y} \end{bmatrix} \ominus \mathcal{D}$ and $\begin{bmatrix} \mathcal{X}' \\ \mathcal{U} \end{bmatrix} \ominus \mathcal{R}$ have the same dimension. We may also assume that $\tilde{\mathcal{X}}$ is equipped with some representation $\tilde{\rho}$ of $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$ and hence \mathcal{X}' is equipped with the representation $\rho' = \rho \oplus \tilde{\rho}$. We now assume that all this has been done and drop the prime notation; thus without loss of generality we have $\dim \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \ominus \mathcal{D} = \dim \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \ominus \mathcal{R}$ and \mathcal{X} is equipped with a $*$ -representation ρ of $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$.

We now let V_0 be any unitary transformation from $\begin{bmatrix} C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \ominus \mathcal{D}$ onto $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \ominus \mathcal{R}$ and set

$$\begin{aligned} \mathbf{U}^* &= V \oplus V_0: \begin{bmatrix} C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \cong \mathcal{D} \oplus \left(\begin{bmatrix} C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \ominus \mathcal{D} \right) \\ &\rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \cong \mathcal{R} \oplus \left(\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \ominus \mathcal{R} \right). \end{aligned}$$

We may then write out \mathbf{U}^* as a block 2×2 -matrix

$$\mathbf{U} = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}: \begin{bmatrix} C_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}.$$

Since \mathbf{U}^* is an extension of V given by (3.16), we have

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} L_{\mathbb{E}(w)^*} H(w)^* y_w \\ y_w \end{bmatrix} = \begin{bmatrix} H(w)^* y_w \\ S_0(w)^* y_w \end{bmatrix}. \quad (3.17)$$

The first row of (3.17) gives

$$A^* L_{\mathbb{E}(w)^*} H(w)^* y_w + C^* y_w = H(w)^* y_w.$$

Since $\sup_{\psi} \{\|\psi(w)\|\} < 1$ by the assumption (2.1) and since $\|A^*\| \leq 1$ as \mathbf{U} is unitary, we see that $I - A^* L_{\mathbb{E}(w)^*}$ is invertible and, by the arbitrariness of $y_w \in \mathcal{Y}$, we can solve (3.17) to get

$$H(w)^* = (I - A^* L_{\mathbb{E}(w)^*})^{-1} C^*.$$

Plugging this into the second row of (3.17) then gives

$$B^* L_{\mathbb{E}(w)^*} (I - A^* L_{\mathbb{E}(w)^*})^{-1} C^* + D^* = S_0(w)^*.$$

Taking adjoints and replacing w by $z \in \Omega_0$ leads to the realization formula (3.5).

We actually get a little bit more. The right-hand side of (3.5) makes sense for z equal to any point in Ω . Thus we have actually proved: (2) \Rightarrow (3') where the precise statement of (3') is:

(3') *There is a Ψ -unitary colligation \mathbf{U} as in (3.4) such that S_0 has an extension to an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued function S defined on all of Ω having the transfer-function realization*

$$S(z) = D + C(I - L_{\mathbb{E}(z)^*}^* A)^{-1} L_{\mathbb{E}(z)^*}^* B \quad (3.18)$$

for $z \in \Omega$.

□

Proof of (3) \Rightarrow (2). We assume that we have a transfer-function realization (3.5) and we must produce a completely positive kernel Γ so that (3.3) holds. There is a natural candidate, namely:

$$\Gamma(z, w)[g] = C(I - L_{\mathbb{E}(z)^*}^* A)^{-1} \rho(g)(I - A^* L_{\mathbb{E}(w)^*})^{-1} C^*. \quad (3.19)$$

The candidate is certainly a completely positive kernel since the formula (3.19) exhibits its Kolmogorov decomposition (condition (3') in Theorem 2.6 with $H(z) = C(I - L_{\mathbb{E}(z)^*}^* A)^{-1}$ and $\pi = \rho$). The verification of (3.3) amounts to the identity

$$I - S_0(z)S_0(w)^* = C(I - L_{\mathbb{E}(z)^*}^* A)^{-1} \rho(I - \mathbb{E}(z)\mathbb{E}(w)^*)(I - A^* L_{\mathbb{E}(w)^*})^{-1} C^*. \quad (3.20)$$

Using the realization formula (3.5) for $S_0(z)$ and the relations

$$AA^* + BB^* = I, \quad AC^* + BD^* = 0, \quad CC^* + DD^* = I$$

coming out of the coisometric property $\mathbf{U}\mathbf{U}^* = I$ of \mathbf{U} then give us

$$\begin{aligned} & I - S_0(z)S_0(w)^* \\ &= I - [D + C(I - L_{\mathbb{E}(z)^*}^* A)^{-1} L_{\mathbb{E}(z)^*}^* B][D^* + B^* L_{\mathbb{E}(w)^*} (I - A^* L_{\mathbb{E}(w)^*})^{-1} C^*] \\ &= I - DD^* - C(I - L_{\mathbb{E}(z)^*}^* A)^{-1} L_{\mathbb{E}(z)^*}^* B D^* - D B^* L_{\mathbb{E}(w)^*} (I - A^* L_{\mathbb{E}(w)^*})^{-1} C^* \\ &\quad - C(I - L_{\mathbb{E}(z)^*}^* A)^{-1} L_{\mathbb{E}(z)^*}^* B B^* L_{\mathbb{E}(w)^*} (I - A^* L_{\mathbb{E}(w)^*})^{-1} C^* \\ &= CC^* + C(I - L_{\mathbb{E}(z)^*}^* A)^{-1} L_{\mathbb{E}(z)^*}^* A C^* + C A^* L_{\mathbb{E}(w)^*} (I - A^* L_{\mathbb{E}(w)^*})^{-1} C^* \\ &\quad + C(I - L_{\mathbb{E}(z)^*}^* A)^{-1} L_{\mathbb{E}(z)^*}^* (A A^* - I) L_{\mathbb{E}(w)^*} (I - A^* L_{\mathbb{E}(w)^*})^{-1} C^* \\ &= C(I - L_{\mathbb{E}(z)^*}^* A)^{-1} X (I - A^* L_{\mathbb{E}(w)^*})^{-1} C^* \end{aligned} \quad (3.21)$$

where we have set X equal to

$$\begin{aligned} X &= (I - L_{\mathbb{E}(z)^*}^* A)(I - A^* L_{\mathbb{E}(w)^*}) + L_{\mathbb{E}(z)^*}^* A(I - A^* L_{\mathbb{E}(w)^*}) \\ &\quad + (I - L_{\mathbb{E}(z)^*}^* A) A^* L_{\mathbb{E}(w)^*} + L_{\mathbb{E}(z)^*}^* A A^* L_{\mathbb{E}(w)^*} - L_{\mathbb{E}(z)^*}^* L_{\mathbb{E}(w)^*} \\ &= I - L_{\mathbb{E}(z)^*}^* L_{\mathbb{E}(w)^*} \\ &= \rho(I - \mathbb{E}(z)\mathbb{E}(w)^*) \end{aligned} \quad (3.22)$$

where we used (2.13) for the last step. Combining (3.21) and (3.22) gives us (3.20) as required. □

Proof of (2) \Rightarrow (1) if $\dim \mathcal{Y}_T < \infty$. We assume that we have an Agler decomposition (3.3) and must show that S_0 can be extended to an S defined on all of Ω which is in the Schur-Agler class $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$. Toward this end, we note that the proof of (2) \Rightarrow (3) really proved (3'), i.e., that S_0 extends to an S defined on all of Ω given by the realization formula (3.18). Therefore the argument behind (3) \Rightarrow (2)

actually gives us an Agler decomposition (3.3) valid for the extended S which holds for z, w in all of Ω . In this way we may assume that S is given to us defined on all of Ω and we are given the completely positive kernel Γ on all of Ω giving rise to the Agler decomposition (3.3) for S .

To check that S is in the Schur-Agler class $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$, we must verify that the operator R_S of right multiplication by S is contractive from $\mathcal{H}(K)_\mathcal{Y}$ to $\mathcal{H}(K)_\mathcal{U}$ for any choice of admissible kernel $K \in \mathcal{K}_\Psi(\mathcal{Y})$. Toward this end, we reverse the procedure used in the proof of (1) \Rightarrow (2) as follows.

Given an admissible kernel $K \in \mathcal{K}_\Psi$ and given any finite collection of points $z_1, \dots, z_N \in \Omega$, we must show that the kernel (3.2) is a positive kernel for all choices of functions $Y: \{z_1, \dots, z_N\} \rightarrow \mathcal{C}_2(\mathcal{Y}, \mathcal{U})$. It suffices to consider the restriction K_0 of K to the finite set $\Omega_0 = \{z_1, \dots, z_N\}$. Since $K \in \mathcal{K}_\Psi(\mathcal{Y})$, we know that the right multiplication operator R_ψ is contractive from $\mathcal{H}(K_0)_{\mathcal{Y}_T}$ to $\mathcal{H}(K_0)_{\mathcal{U}_T}$ for each $\psi \in \Psi$. Consider the modified kernel

$$K_{0,\epsilon}(z, w) = K_0(z, w) + \epsilon^2 \sum_{z \in \Omega_0} \delta_{z,w} I_\mathcal{Y}$$

where $\delta_{z,w}$ is the Kronecker delta function equal to 1 for $z = w$ and 0 otherwise. Since the values of ψ are contractive, we see that R_ψ is still contractive as an operator from $\mathcal{H}(K_{0,\epsilon})_{\mathcal{Y}_T}$ to $\mathcal{H}(K_{0,\epsilon})_{\mathcal{U}_T}$ for each $\epsilon > 0$. Also, to show that R_S is contractive from $\mathcal{H}(K_0)_\mathcal{Y}$ to $\mathcal{H}(K_0)_\mathcal{U}$, it is enough to show that R_S is contractive from $\mathcal{H}(K_{0,\epsilon})_\mathcal{Y}$ to $\mathcal{H}(K_{0,\epsilon})_\mathcal{U}$ for each $\epsilon > 0$.

Our next goal is to construct a kernel $L_\epsilon: \Omega_0 \times \Omega_0 \rightarrow \mathcal{L}(\mathcal{Y})$ so that

$$\langle f, g \rangle_{\mathcal{H}(K_{0,\epsilon})} = \sum_{z, w \in \Omega_0} \text{tr}(L_\epsilon(z, w) f(z) g(w)^*). \quad (3.23)$$

To do this, define $L(z, w) \in \mathcal{L}(\mathcal{Y})$ by

$$\langle L_\epsilon(z, w) u, v \rangle_\mathcal{Y} = \langle \delta_z u, \delta_w v \rangle_{\mathcal{H}(K_{0,\epsilon})}$$

where δ_z is the point-mass function

$$\delta_z(z') = \begin{cases} 1 & \text{if } z = z', \\ 0 & \text{otherwise.} \end{cases}$$

In terms of the kernel function $K_{0,\epsilon}$, one can verify the block-matrix identity

$$[L_\epsilon(z, w)]_{z, w, \in \Omega_0} = ([K_{0,\epsilon}(z, w)]_{z, w \in \Omega_0})^{-1}.$$

The fact that $R_\psi: \mathcal{H}(K_{0,\epsilon})_{\mathcal{Y}_T} \rightarrow \mathcal{H}(K_{0,\epsilon})_{\mathcal{U}_T}$ is contractive can be equivalently expressed as

$$\sum_{z, w, \in \Omega_0} \text{tr}(L_\epsilon(z, w) f(z) (I - \psi(z) \psi(w)^*) f(w)^*) \geq 0 \text{ for all } f: \Omega \rightarrow \mathcal{C}_2(\mathcal{Y}_T, \mathcal{Y}). \quad (3.24)$$

To show that $R_S: \mathcal{H}(K_{0,\epsilon})_\mathcal{Y} \rightarrow \mathcal{H}(K_{0,\epsilon})_\mathcal{U}$ is contractive can be expressed in a similar way as

$$\sum_{z, w \in \Omega_0} \text{tr}(L_\epsilon(z, w) h(z) (I - S(z) S(w)^*) h(w)^*) \geq 0 \text{ for all } h: \Omega_0 \rightarrow \mathcal{C}_2(\mathcal{Y}). \quad (3.25)$$

By assumption we are given an Agler decomposition (3.3) for S . The completely positive kernel Γ appearing in (3.3) in turn has a Kolmogorov decomposition as in

(3') in Theorem 2.6:

$$\Gamma(z, w)[g] = H(z)\rho(g)H(w)^* \quad (3.26)$$

for a $*$ -representation $\rho: C_b(\Psi, \mathcal{L}(\mathcal{Y}_T)) \rightarrow \mathcal{L}(\mathcal{X})$. We now use the assumption that $\dim \mathcal{Y}_T < \infty$. This has the effect that $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$ is a CCR C^* -algebra and that any representation ρ of $C_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$ is the direct integral of multiples of irreducible representations, where an irreducible representation $\pi_0: C(\Psi_\beta, \mathcal{L}(\mathcal{Y}_T)) \rightarrow \mathcal{L}(\mathcal{Y}_T)$ has the point-evaluation form $\pi_0(g) = g(\psi_0)$ for some $\psi_0 \in \Psi_\beta$; we refer to [13] and [35, Section 2.3] for fuller discussion. Thus we may assume that there are mutually singular measures $\mu_\infty, \mu_1, \mu_2, \dots$ defined on the Borel subsets of the Stone-Ćech compactification Ψ_β of Ψ so that

$$\rho = \infty \cdot \pi_{\mu_\infty} \oplus 1 \cdot \pi_{\mu_1} \oplus 2 \cdot \pi_{\mu_2} \oplus \dots$$

where

$$\pi_{\mu_j}(g): f(\psi) \mapsto g(\psi)f(\psi) \text{ on } \mathcal{H}_{\pi_j} := L^2_{\mathcal{Y}_T}(\mu_j) = L^2(\mu_j) \otimes \mathcal{Y}_T$$

and where in general $n \cdot \pi$ refers to the n -fold inflation of π :

$$(n \cdot \pi)(g) = \begin{bmatrix} \pi(g) & & \\ & \ddots & \\ & & \pi(g) \end{bmatrix} \text{ on } (\mathcal{H}_\pi)^n := \bigoplus_{j=1}^n \mathcal{H}_\pi.$$

Thus we may assume that the representation space \mathcal{X} in (3.26) decomposes as

$$\mathcal{X} = L^2_{\mathcal{Y}_T}(\mu_\infty)^\infty \oplus \bigoplus_{r=1}^\infty L^2_{\mathcal{Y}_T}(\mu_r)^r.$$

Therefore the operators $H(w)^*$ appearing in (3.26) decompose as

$$H(w)^* = \begin{bmatrix} H_\infty(w)^* \\ \text{col}_{r=1}^\infty H_r(w)^* \end{bmatrix}$$

where each $H_r(w)^*$ is an operator from \mathcal{Y} to $L^2_{\mathcal{Y}_T}(\mu_r)^r$. This enables us to define an operator-valued function $H_r(w, \psi)^*$ of $\psi \in \Psi_\beta$ according to

$$H_r(w, \psi)^* y = ((H_r(w)^* y)(\psi)).$$

Then the adjoint $H_r(z)$ of $H_r(z)^*$ is given via an integral formula:

$$H_r(z): G(\psi) \mapsto \int_{\Psi_\beta} H_r(z, \psi) G(\psi) d\mu_r(\psi).$$

We conclude that the Agler decomposition (3.3) takes the more detailed form

$$\begin{aligned} I - S(z)S(w)^* &= \int_{\Psi_\beta} H_\infty(z, \psi) (I_{\ell^2} \otimes (I - \psi(z)\psi(w)^*)) H_\infty(w, \psi)^* d\mu_\infty(\psi) \\ &\quad + \sum_{r=1}^\infty \int_{\Psi_\beta} H_r(z, \psi) (I_{\mathbb{C}^r} \otimes (I - \psi(z)\psi(w)^*)) H_r(w, \psi)^* d\mu_r(\psi). \end{aligned} \quad (3.27)$$

Plugging this into the left-hand side of the desired inequality in (3.25) and taking the integral to the outside gives us the sum over $z, w \in \Omega_0$ of the following terms:

$$\int_{\Psi_\beta} \operatorname{tr} (L_\epsilon(z, w) h(z) H_\infty(z, \psi) (I_{\ell^2} \otimes (I - \psi(z) \psi(w)^*)) H_\infty(w, \psi)^* h(w)^*) d\mu_\infty(\psi) + \sum_{r=1}^{\infty} \int_{\Psi_\beta} \operatorname{tr} (L_\epsilon(z, w) h(z) H_r(z, \psi) (I_{\mathbb{C}^r} \otimes (I - \psi(z) \psi(w)^*)) H_r(w, \psi)^* h(w)^*) d\mu_r(\psi).$$

From (3.24) we see that the sum over $z, w \in \Omega_0$ of the integrand in each of these terms is nonnegative. Hence the sum over z, w of the integrals in nonnegative and (3.25) follows as required. \square

Remark 3.7. The interpolation problem for the class $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$ can be formulated as follows: *Given a subset Ω_0 of Ω and a function $S_0: \Omega_0 \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$, give necessary and sufficient conditions for the existence of an $S \in \mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$ such that $S|_{\Omega_0} = S_0$.* Assuming that $\dim \mathcal{Y}_T < \infty$, one gets a solution criterion (arguably not particularly practical at this level of generality) immediately from the equivalence (1) \Leftrightarrow (2) in Theorem 3.1 (where we use (2) in the more concrete form (3.27)): *the $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$ -interpolation problem has a solution if and only if there exists a matrix-valued function $(\psi, z) \mapsto H_\psi(z)$ on $\Psi_\beta \times \Omega_0$, bounded and measurable in ψ for each z , together with a finite measure μ on Ψ_β , so that*

$$I - S_0(z) S_0(w)^* = \int_{\Psi_\beta} H_\psi(z) (I_{\mathcal{X}_\psi} \otimes (I - \psi(z) \psi(w)^*)) H_\psi(w)^* d\mu(\psi)$$

for each $z, w \in \Omega_0$. Not so apparent from the way Theorem 3.1 is formulated is that condition (1) by itself is also a criterion for solving the interpolation problem. Indeed, if we set $\Psi|_{\Omega_0}$ equal to the collection of restricted functions

$$\Psi|_{\Omega_0} = \{\psi|_{\Omega_0} : \psi \in \Psi\}, \quad (3.28)$$

we may view $\Psi|_{\Omega_0}$ as itself a collection of test functions generating a Schur-Agler class $\mathcal{SA}_{\Psi|_{\Omega_0}}(\mathcal{U}, \mathcal{Y})$ of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued functions defined only on Ω_0 . The only part of the hypothesis that S_0 extends to an $S \in \mathcal{SA}_\Psi$ used to prove (1) \Rightarrow (2) in Theorem 3.1 is that then $S_0 \in \mathcal{SA}_{\Psi|_{\Omega_0}}$. We conclude that we get another criterion for solution of the interpolation problem: *the $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$ -interpolation problem has a solution if and only if $S_0 \in \mathcal{SA}_{\Psi|_{\Omega_0}}$.* Let us say that the subset $\mathcal{K}_\Psi^0(\mathcal{Y})$ of the set of admissible kernels $\mathcal{K}_\Psi(\mathcal{Y})$ is a *generating set* for $\mathcal{K}_\Psi(\mathcal{Y})$ if, for each kernel $K \in \mathcal{K}_\Psi(\mathcal{Y})$, there is a kernel $K^0 \in \mathcal{K}_\Psi^0(\mathcal{Y})$ such that K is congruent to K^0 in the sense that there is an operator function Y so that $K(z, w) = Y(z) K^0(z, w) Y(w)^*$. It is easy to check that the kernels of the form (3.2) are positive on Ω_0 for all Y and admissible K if and only if all such kernels are positive when the admissible K is restricted to those coming from the generating set $\mathcal{K}_\Psi^0(\mathcal{Y})$. Hence we arrive at the following dual criterion for solution of the $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$ -interpolation problem: *the $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$ -interpolation problem has a solution if and only if the kernel*

$$k(z, w) = \operatorname{tr} (Y(w)^* (I - S_0(w)^* S_0(z)) Y(z) K^0(z, w))$$

is a positive kernel on Ω_0 for all $Y: \Omega_0 \rightarrow \mathcal{C}_2(\mathcal{Y}, \mathcal{U})$ for all admissible kernels K from the generating set $\mathcal{K}_\Psi^0(\mathcal{Y})$. We illustrate these ideas on the examples discussed in Section 4 below. This duality pairing between admissible kernels and test functions is central to the operator-algebra point of view of Paulsen and Solazzo toward interpolation theory (see [45, 46, 47]).

There is also an operator-algebra point of view toward the Schur-Agler class. For convenience in the following discussion, we take all the coefficient spaces \mathcal{U} , \mathcal{Y} , \mathcal{U}_T , and \mathcal{Y}_T to be the same space \mathcal{U} although this probably is not essential. We abbreviate the notation $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{U})$ to $\mathcal{SA}_\Psi(\mathcal{U})$. Let $\Psi|\Omega_0$ be as in (3.28) and let $H_{\Psi|\Omega_0}^\infty(\mathcal{U})$ denote the space of all $\mathcal{L}(\mathcal{U})$ -valued functions S_0 on the subset Ω_0 of Ω such that there exists a positive $M < \infty$ so that the kernel $k_{X, S_0, K, M}$ given by (2.9) is a positive kernel on Ω_0 for all choices of $X: \Omega_0 \rightarrow \mathcal{C}_2(\mathcal{E}, \mathcal{U})$ and for all choices of K for which the kernel $k_{Y, \psi, K, 1}$ is positive for all choices of $Y: \Omega_0 \rightarrow \mathcal{C}_2(\mathcal{U})$ and $\psi \in \Psi$, or, what is the same, such that the right multiplication operator R_S has norm at most M as an operator on $\mathcal{H}(K)_\mathcal{U}$ for all positive kernels K for which R_ψ has norm at most 1 on $\mathcal{H}(K)_\mathcal{U}$ for all $\psi \in \Psi$. We define the $H_{\Psi|\Omega_0}^\infty$ -norm $\|S\|_{H_{\Psi|\Omega_0}^\infty}$ as the infimum of all such positive numbers M . Then $H_{\Psi|\Omega_0}^\infty(\mathcal{U})$ is an operator algebra with unit ball equal to the Schur-Agler class $\mathcal{SA}_{\Psi|\Omega_0}(\mathcal{U})$. The following representation-theoretic characterization of the Schur-Agler class will be convenient in Section 4.1 below.

Theorem 3.8. *Suppose that Ψ , $\Omega_0 \subset \Omega$, and S_0 are as in Theorem 3.1 with $\mathcal{U} = \mathcal{Y} = \mathcal{U}_T = \mathcal{Y}_T$. In addition to conditions (1), (2), (3) in Theorem 3.1, consider:*

- (4) *For any representation $\pi: H_{\Psi|\Omega_0}^\infty(\mathcal{U}) \rightarrow \mathcal{L}(\mathcal{K})$ such that $\|\pi(\psi)\| \leq 1$ for all $\psi \in \Psi$, it also holds that $\|\pi(S_0)\| \leq 1$.*

Then (4) \Rightarrow (1). If $\dim \mathcal{U} < \infty$, then also (2) \Rightarrow (4) and (1), (2), (3), and (4) are all equivalent.

Proof. Assume (4) holds and suppose that $K \in \mathcal{K}_{\Psi|\Omega_0}(\mathcal{U})$ is an admissible kernel. We now view the map $\pi_K: H_{\Psi|\Omega_0}^\infty(\mathcal{U}) \rightarrow \mathcal{L}(\mathcal{H}(K)_\mathcal{U})$ sending $G \in H_{\Psi|\Omega_0}^\infty(\mathcal{U})$ to the right multiplication operator R_G on $\mathcal{H}(K)_\mathcal{U}$ as a representation (technically, an anti-representation, but this does not affect the final results). By definition of $K \in \mathcal{K}_{\Psi|\Omega_0}(\mathcal{U})$, we have $\|\pi_K(\psi)\| \leq 1$ for each $\psi \in \Psi$. Condition (4) then tells us that $\|\pi(S_0)\| \leq 1$, i.e., R_{S_0} on $\mathcal{H}(K)_\mathcal{U}$ has norm at most 1. In this way we have verified condition (1).

Conversely, we suppose $\dim \mathcal{Y}_T = \dim \mathcal{U} < \infty$ and that condition (2) holds. As in the proof of (2) \Rightarrow (1) we see that (2) can be written in the more explicit form (3.27). Given any $\mathcal{L}(\mathcal{U})$ -valued kernel $\mathbf{K}(z, w)$ with a factorization $\mathbf{K}(z, w) = F(z)G(w)^*$ with $F, G \in H_{\Psi|\Omega_0}^\infty(\mathcal{U})$, we use the hereditary functional calculus to extend a given representation π of $H_{\Psi|\Omega_0}^\infty(\mathcal{U})$ to such kernels according to the rule

$$\pi(F(z)G(w)^*) = \pi(F)\pi(G)^*.$$

Applying π to (3.27) (and using continuity to push π past the integral sign) gives

$$\begin{aligned} I - \pi(S_0)\pi(S_0)^* &= \int_{\Psi_\beta} \pi(H_\infty(\cdot, \psi)) (I_{\ell^2} \otimes (I - \pi(\psi)\pi(\psi)^*)) \pi(H_\infty(\cdot, \psi))^* d\mu_\infty(t) \\ &\quad + \sum_{r=1}^{\infty} \int_{\Psi_\beta} \pi(H_r(\cdot, \psi)) (I_{\mathbb{C}^r} \otimes (I - \pi(\psi)\pi(\psi)^*)) \pi(H_r(\cdot, \psi))^* d\mu_r(\psi). \end{aligned}$$

From the fact that $\|\pi(\psi)\| \leq 1$ for each $\psi \in \Psi$ we read off from this last expression that $\|\pi(S_0)\| \leq 1$ as well, i.e., (4) is verified. \square

Remark 3.9. In the proof of Theorem 3.1 we drew on a lot of ideas which have been used in previous versions of this type of result, starting with the seminal

paper of Agler [3] and continuing with [5, 22, 10, 23, 32, 53, 16, 19, 8, 27, 29] as well as commutant lifting versions [23, 21, 9, 41]. In particular, the cone separation argument in the proof of $(1) \Rightarrow (2)$ and the proof of $(2) \Rightarrow (3)$ (the so-called lurking-isometry argument) go back to [3]. However there are some new technical difficulties in the test-function setting where some new ideas are required in order to arrive at the final result; we now discuss some of these.

In the proof of $(1) \Rightarrow (2)$, the use of the ϵ^2 -perturbation term in the definition of the $\mathcal{H}_{\mathbb{L}_1, \epsilon}$ norm is the ploy needed to make the point-evaluations $f \mapsto f(w)$ bounded and enables us to avoid the hypothesis that the set of test functions Ψ separates the points of any finite subset Ω_F of Ω , as used in [27, 29].

Our proof of $(2) \Rightarrow (1)$ (with the hypothesis that $\dim \mathcal{Y}_T < \infty$) is close to the proof of $(3) \Rightarrow (1)$ in [29] (for the scalar-valued case) (which actually involves use of the representation-theory formulation (4)). These authors make use of the spectral theorem for a representation of $C_b(\Psi, \mathbb{C})$, approximating a general representation ρ by a “simple representation” (approximation of the general integral in (3.27) by a simple-function integrand). Thus their proof also makes use of the CCR character of $C_b(\Psi, \mathbb{C})$, and hence does not appear to extend to the case $\dim \mathcal{Y}_T = \infty$.

4. ALGEBRAS ARISING FROM TEST FUNCTIONS

In this section, rather than starting with a set of test functions Ψ , we assume that we are given a function algebra \mathcal{A} and then seek to determine a set of test functions $\Psi_{\mathcal{U}, \mathcal{Y}}$ so that the unit ball of the operator-valued version of \mathcal{A} , say $\mathcal{A} \otimes \mathcal{L}(\mathcal{U}, \mathcal{Y})$ where \mathcal{U}, \mathcal{Y} are two coefficient Hilbert spaces, can be identified as the associated Schur-Agler class $\mathcal{SA}_{\Psi_{\mathcal{U}, \mathcal{Y}}}(\mathcal{U}, \mathcal{Y})$.

The classical example is the Hardy algebra over the unit disk $\mathcal{A} = H^\infty(\mathbb{D})$. The operator-valued version $\mathcal{A} \otimes \mathcal{L}(\mathcal{U}, \mathcal{Y})$ has unit ball equal to the classical operator-valued Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$, for which we have the now classical result: $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ if and only if the associated de Branges-Rovnyak kernel $K_S(z, w) = [I - S(z)S(w)^*]/(1 - z\bar{w})$ is a positive kernel on \mathbb{D} . If we let $K_S(z, w) = H(z)H(w)^*$ be the Kolmogorov decomposition of K_S , then we arrive at

$$I - S(z)S(w)^* = H(z)((1 - z\bar{w})I_{\mathcal{X}})H(w)^*$$

which is exactly the Agler decomposition (3.3) corresponding to the singleton collection of test functions $\Psi = \{\psi_0\}$ with ψ_0 equal to the coordinate function: $\psi_0(z) = z$. For this case, moving from the scalar-valued case to the matrix- or operator-valued case necessitates no change in the choice of test-function set Ψ . A similar story holds for the case of the Schur-Agler class over the polydisk [22], the Schur-multiplier class over the Drury-Arveson space [23, 32], and the Schur-Agler class over more general domains in \mathbb{D}^d with matrix polynomial or analytic defining function [16, 9]. However the situation for the case where \mathcal{A} is the algebra of bounded analytic functions over a finitely connected planar domain \mathcal{R} , or where \mathcal{A} is the constrained Hardy algebra over the unit disk (bounded holomorphic functions f on \mathbb{D} with the extra constraint that $f'(0) = 0$) is quite different. We discuss each of these in turn.

4.1. The Schur class over a multiply connected planar domain. We let \mathcal{R} denote a bounded domain (connected, open set) in the complex plane \mathbb{C} whose boundary consists of $m + 1$ smooth Jordan curves $\partial_0, \partial_1, \dots, \partial_m$ with ∂_0 denoting the boundary of the unbounded component of the complement of \mathcal{R} in \mathbb{C} . We let $\mathcal{S}_{\mathcal{R}}$ denote the space of holomorphic functions mapping \mathcal{R} into the unit disk,

and $\mathcal{S}_{\mathcal{R}}(\mathcal{U}, \mathcal{Y})$ the operator-valued version consisting of holomorphic functions on \mathcal{R} with values in the closed unit ball $\overline{\mathcal{BL}}(\mathcal{U}, \mathcal{Y})$ of bounded linear operators between two coefficient Hilbert spaces \mathcal{U} and \mathcal{Y} . In [28] there was identified a collection of inner functions $\{s_{\mathbf{x}}: \mathbf{x} \in \mathbb{T}_{\mathcal{R}}\}$, normalized to have value 1 at a fixed point $\zeta_0 \in \partial_0$ and to satisfy $s(t_0) = 0$ at a fixed point $t_0 \in \mathcal{R}$, having exactly m zeros in \mathcal{R} (the minimal number possible for a single-valued inner function on \mathcal{R}), and indexed by \mathbf{x} belonging to the \mathcal{R} -torus $\mathbb{T}_{\mathcal{R}} := \partial_0 \times \partial_1 \times \cdots \times \partial_m$, so that any scalar Schur class function $s \in \mathcal{S}_{\mathcal{R}}$ has an Agler decomposition (3.3) with respect to the family $\Psi = \{\psi_{\mathbf{x}}: \mathbf{x} \in \mathbb{T}_{\mathcal{R}}\}$ s in (1.11) (or (3.27) specialized to this case):

$$1 - s(z)\overline{s(w)} = \int_{\mathbb{T}_{\mathcal{R}}} h_{\mathbf{x}}(z) \left(1 - s_{\mathbf{x}}(z)\overline{s_{\mathbf{x}}(w)}\right) \overline{h_{\mathbf{x}}(w)} d\nu(\mathbf{x}). \quad (4.1)$$

In more detail, the functions $s_{\mathbf{x}}$ are constructed as follows. Let $\phi = \{\phi_1, \dots, \phi_m\}$ be real-valued continuous functions on $\partial\mathcal{R}$ such that

$$\{\phi_1, \dots, \phi_m\} = \text{basis for } L^2(\omega_{t_0}) \ominus [H^2(\omega_{t_0}) + \overline{H^2(\omega_{t_0})}] \quad (4.2)$$

where ω_{t_0} is the harmonic measure on $\partial\mathcal{R}$ for some fixed point $t_0 \in \mathcal{R}$ (so $h(t_0) = \int_{\partial\mathcal{R}} h(\zeta) d\omega_{t_0}(\zeta)$ for h harmonic on \mathcal{R} and continuous on \mathcal{R}^-), $H^2(\omega_{t_0})$ is the associated Hardy space, and the overline indicates complex conjugation—see e.g. [33]. Then given $\mathbf{x} = (x_0, x_1, \dots, x_m) \in \mathbb{T}_{\mathcal{R}}$, there is a unique choice of weights $w_0^{\mathbf{x}}, w_1^{\mathbf{x}}, \dots, w_m^{\mathbf{x}}$, each positive with sum equal to 1, so that

$$\sum_{r=0}^m w_r^{\mathbf{x}} \phi_i(x_r) = 0 \text{ for } i = 1, \dots, m \quad (4.3)$$

(see [4, Theorem 3.1.17]). Given any \mathbf{x} and the associated weights $(w_0^{\mathbf{x}}, w_1^{\mathbf{x}}, \dots, w_m^{\mathbf{x}})$ we associate the probability measure on $\partial\mathcal{R}$:

$$\mu_{\mathbf{x}} := \sum_{r=0}^m w_r^{\mathbf{x}} \delta_{x_r}$$

where δ_{x_r} is the unit point-mass measure at x_r . The constraint (4.3) guarantees that the harmonic function

$$h_{\mathbf{x}}(z) = \int_{\partial\mathcal{R}} \mathcal{P}_z(\zeta) d\mu_{\mathbf{x}}(\zeta)$$

(where $\mathcal{P}_z(\zeta)$ is the poisson kernel normalized to have $\mathcal{P}_{t_0}(\zeta) = 1$) has single-valued harmonic conjugate. We then define $f_{\mathbf{x}}(z)$ to be the unique holomorphic function on \mathcal{R} with

$$\operatorname{Re} f_{\mathbf{x}}(z) = h_{\mathbf{x}}(z) \text{ and } f_{\mathbf{x}}(t_0) = 1.$$

Finally we set

$$s_{\mathbf{x}}(z) = \frac{f_{\mathbf{x}}(z) - 1}{f_{\mathbf{x}}(z) + 1}. \quad (4.4)$$

Then $s_{\mathbf{x}}$ are the inner functions appearing in (4.1), apart from the additional normalization that $s_{\mathbf{x}}(\zeta_0) = 1$ at a fixed $\zeta_0 \in \partial_0$. Then it is shown in [29] that $\mathcal{S}_{\mathcal{R}} = \mathcal{SA}_{\Psi_{\mathcal{R}}}$ with the collection of test functions $\Psi_{\mathcal{R}}$ taken to be $\Psi_{\mathcal{R}} = \{s_{\mathbf{x}}: \mathbf{x} \in \mathbb{T}_{\mathcal{R}}\}$. There it is shown, at least for the annulus case ($m = 1$), that, with the additional normalization $s_{\mathbf{x}}(\zeta_0) = 1$ imposed, that $\Psi_{\mathcal{R}}$ is minimal in the sense that no nonempty open subset of $\mathbb{T}_{\mathcal{R}}$ can be omitted and still have the decomposition (4.1) hold for all $s \in \mathcal{S}_{\mathcal{R}}$.

Before explaining the matrix generalization of (4.4), we first recall some ideas from [20]. Suppose that we are given a collection

$$\phi = \left\{ \phi^{(1)} = \begin{bmatrix} \phi_1^{(1)} \\ \vdots \\ \phi_m^{(1)} \end{bmatrix}, \dots, \phi^{(n)} = \begin{bmatrix} \phi_1^{(n)} \\ \vdots \\ \phi_m^{(n)} \end{bmatrix} \right\}$$

of n vectors in \mathbb{R}^m . From ϕ we form the block column vectors

$$\phi \otimes I_N = \left\{ \phi^{(1)} \otimes I_N := \begin{bmatrix} \phi_1^{(1)} I_N \\ \vdots \\ \phi_m^{(1)} I_N \end{bmatrix}, \dots, \phi^{(n)} \otimes I_N := \begin{bmatrix} \phi_1^{(n)} I_N \\ \vdots \\ \phi_m^{(n)} I_N \end{bmatrix} \right\}$$

in $(\mathbb{C}^{N \times N})^m$ ($m \times 1$ -column vectors with entries of size $N \times N$). We then say that

the zero element $\mathbf{0} = \begin{bmatrix} 0_{N \times N} \\ \vdots \\ 0_{N \times N} \end{bmatrix}$ of $(\mathbb{C}^{N \times N})^m$ is *in the C^* -convex hull of $\phi \otimes I_N$* if there exist positive semidefinite $N \times N$ matrices W_1, \dots, W_n with $\sum_{r=1}^n W_r = I_N$ so that

$$\mathbf{0} = \sum_{r=1}^n \phi^{(r)} \otimes W_r \quad (4.5)$$

where we set $\phi^{(r)} \otimes W_r = \begin{bmatrix} \phi_1^{(r)} W_r \\ \vdots \\ \phi_m^{(r)} W_r \end{bmatrix}$. We say that $\mathbf{0}$ is *in the interior of the C^* -convex hull of $\phi \otimes I_N$* if in addition the matrix weights $\{W_1, \dots, W_n\}$ have the property that their range spaces $\{\text{Ran } W_1, \dots, \text{Ran } W_n\}$ are *ϕ -constrained weakly independent* by which we mean: *whenever T_1, \dots, T_n are $N \times N$ complex Hermitian matrices with $\text{Ran } T_r \subset \text{Ran } W_r$ for each $r = 1, \dots, n$ such that*

$$\sum_{r=1}^n T_r = 0 \text{ and } \sum_{r=1}^n \phi_i(x_r) T_r = 0 \text{ for } i = 1, \dots, n,$$

it follows that $T_r = 0$ for each $r = 1, \dots, n$. When all this happens, we refer to $\{W_1, \dots, W_n\}$ as a *choice of matrix barycentric coordinates of $\mathbf{0}$ with respect to ϕ .*

By way of motivation for these notions, note that, in case $N = 1$ and all the weights $W_1 = w_1, \dots, W_n = w_n$ (now complex numbers) are nonzero (which can be arranged simply by discarding appropriate vectors $\phi^{(r)}$ from the list of vectors ϕ), then $\mathbf{0} = 0 \in \mathbb{R}^m$ in the interior of the C^* -convex hull of $\phi \otimes I_1 = \phi$ simply means that the vector $0 \in \mathbb{R}^m$ is in the interior of the simplex generated by the vectors $\phi^{(1)}, \dots, \phi^{(n)}$ and that w_1, \dots, w_n are the classical barycentric coordinates for 0 with respect to the simplex vertices $\phi^{(1)}, \dots, \phi^{(n)}$.

We are now ready to explain the matrix analogue of the \mathcal{R} -torus $\mathbb{T}_{\mathcal{R}}$ used to parametrize the set of scalar test functions (4.4). We define the matrix \mathcal{R} -torus $\mathbb{T}_{\mathcal{R}}^N$ to consist of all pairs (\mathbf{x}, \mathbf{w}) of the form $(\mathbf{x}, \mathbf{w}) = (x_1, \dots, x_n; W_1, \dots, W_n)$ where x_1, \dots, x_n is a set of n distinct points in $\partial \mathcal{R}$ such that $\mathbf{0}$ is in the interior of the C^* -convex hull of the set of vectors $\phi(\mathbf{x}) \otimes I_N$, where we set

$$\phi(\mathbf{x}) = \left\{ \phi(x_1) = \begin{bmatrix} \phi_1(x_1) \\ \vdots \\ \phi_m(x_1) \end{bmatrix}, \dots, \phi(x_n) = \begin{bmatrix} \phi_1(x_n) \\ \vdots \\ \phi_m(x_n) \end{bmatrix} \right\}, \quad (4.6)$$

with ϕ_1, \dots, ϕ_m as in (4.2), and with $\{W_1, \dots, W_n\}$ is a choice of matrix barycentric coordinates for $\mathbf{0}$ with respect to $\phi(\mathbf{x}) \otimes I_N$. In particular, the condition (4.5) in the present context specializes to

$$\sum_{r=1}^n \phi_i(x_r) W_r = 0 \text{ for } i = 1, \dots, m. \quad (4.7)$$

For the case $N = 1$, necessarily $n = m + 1$, after a reindexing the collection of points (x_0, x_1, \dots, x_m) necessarily consists of exactly one point from each boundary component $\partial_0, \dots, \partial_m$, and the associated scalar weights $w_0^{\mathbf{x}}, w_1^{\mathbf{x}}, \dots, w_m^{\mathbf{x}}$ are uniquely determined by \mathbf{x} . For $N > 1$, the characterization of $\mathbb{T}_{\mathcal{R}}^N$ is not so explicit; nevertheless it is nonempty and is a well-defined metrizable topological space which is in one-to-one correspondence with a collection of quantum measures (positive matrix measures with total mass equal to the identity matrix I_N) which we define next. For additional information we refer to [20].

Given $(\mathbf{x}, \mathbf{w}) \in \mathbb{T}_{\mathcal{R}}^N$, we associate a quantum measure $\mu_{\mathbf{x}, \mathbf{w}}$ by

$$\mu_{\mathbf{x}, \mathbf{w}} = \sum_{r=1}^n W_r \delta_{x_r} \text{ if } (\mathbf{x}, \mathbf{w}) = (x_1, \dots, x_n; W_1, \dots, W_n) \in \mathbb{T}_{\mathcal{R}}^N. \quad (4.8)$$

Then a consequence of (4.7) is that the matrix-valued harmonic function

$$H_{\mathbf{x}, \mathbf{w}}(z) = \int_{\partial \mathcal{R}} \mathcal{P}_z(\zeta) d\mu_{\mathbf{x}, \mathbf{w}}(\zeta)$$

has a single-valued (matrix-valued) harmonic conjugate, and hence there is a uniquely determined holomorphic function $F_{\mathbf{x}, \mathbf{w}}$ on \mathcal{R} with

$$\operatorname{Re} F_{\mathbf{x}, \mathbf{w}}(z) = H_{\mathbf{x}, \mathbf{w}}(z) \text{ and } F_{\mathbf{x}, \mathbf{w}}(t_0) = I_N.$$

It can be shown that the collection of functions

$$\{F_{\mathbf{x}, \mathbf{w}} : (\mathbf{x}, \mathbf{w}) \in \mathbb{T}_{\mathcal{R}}^N\} \quad (4.9)$$

is exactly the set of extreme points for the compact convex set $\mathcal{H}^N(\mathcal{R})_I$ of normalized Herglotz functions over \mathcal{R} given by

$$\mathcal{H}^N(\mathcal{R})_I = \{F : \mathcal{R} \mapsto \mathbb{C}^{N \times N} : F \text{ holomorphic, } \operatorname{Re} F(z) \geq 0 \text{ for } z \in \mathcal{R}, F(t_0) = I_N\}.$$

Finally, we set

$$S_{\mathbf{x}, \mathbf{w}}(z) = (F_{\mathbf{x}, \mathbf{w}}(z) + I)^{-1} (F_{\mathbf{x}, \mathbf{w}}(z) - I). \quad (4.10)$$

Note that each $S_{\mathbf{x}, \mathbf{w}}(z)$ is an $N \times N$ matrix inner function on \mathcal{R} normalized to satisfy $S(t_0) = 0$. Then in [20] it is shown that any matrix-valued function S in the Schur class $\mathcal{S}_{\mathcal{R}}(\mathbb{C}^N, \mathbb{C}^N)$ has an Agler decomposition of the form

$$I - S(z)S(w)^* = \int_{\mathbb{T}_{\mathcal{R}}^N} H_{\mathbf{x}, \mathbf{w}}(z) (I - S_{\mathbf{x}, \mathbf{w}}(z)S_{\mathbf{x}, \mathbf{w}}(w)^*) H_{\mathbf{x}, \mathbf{w}}(w)^* d\nu(\mathbf{x}, \mathbf{w}) \quad (4.11)$$

for appropriate matrix functions $H_{\mathbf{x}, \mathbf{w}}(z)$ and probability measure ν on $\mathbb{T}_{\mathcal{R}}^N$.

Following the arguments in [29] (adapted to the matrix-valued setting) leads to the following identification of the matrix Schur class $\mathcal{S}_{\mathcal{R}}(\mathbb{C}^N)$ with a matrix-valued test-function Schur class $\mathcal{SA}_{\Psi_{\mathcal{R}}^N}$; the main ingredients of the proof also appear in the more involved proof of Theorem 4.4 below.

Theorem 4.1. *Let $\Psi_{\mathcal{R}}^N$ be the collection of matrix inner functions*

$$\Psi_{\mathcal{R}}^N = \{S_{\mathbf{x}, \mathbf{w}} : (\mathbf{x}, \mathbf{w}) \in \mathbb{T}_{\mathcal{R}}^N\} \quad (4.12)$$

with $S_{\mathbf{x}, \mathbf{w}}$ as in (4.10), with the additional normalization $S_{\mathbf{x}, \mathbf{w}}(\zeta_0) = I_N$ at some fixed point $\zeta_0 \in \partial_0$. Then the matrix-valued Schur class $\mathcal{S}_{\mathcal{R}}(\mathbb{C}^N)$ is identical to the matrix-valued test-function Schur-Agler class $\mathcal{SA}_{\Psi_{\mathcal{R}}^N}$ associated with the collection of test functions $\Psi_{\mathcal{R}}^N$ (as defined by (3.1) and (3.2)).

Combined with Theorem 3.1 and Remark 3.7, we arrive at the following dual formulations of interpolation criteria for the Nevanlinna-Pick interpolation problem for the matrix Schur class over \mathcal{R} . Before stating the result we need a little more background concerning function theory on \mathcal{R} . There is a standard procedure (see e.g. [1]) for introducing m disjoint simple curves $\gamma_1, \dots, \gamma_m$ so that $\mathcal{R} \setminus \gamma$ (where we set γ equal to the union $\gamma = \gamma_1 \cup \dots \cup \gamma_m$) is simply connected. For each cut γ_r we assign some orientation, so that points z not on γ_r but in a sufficiently small neighborhood of γ_r in \mathcal{R} can be assigned a location of either “to the left” or “to the right”. For f a vector-valued function on \mathcal{R} and z a point on some γ_r , we let $f(z_+)$ denote the limit of $f(\zeta)$ as ζ approaches z from the right of γ_r in \mathcal{R} , and similarly, $f(z_-)$ the limit of $f(\zeta)$ as ζ approaches z from the left of γ_r in \mathcal{R} , whenever these limits exist. Given a $\mathbf{U} = (U_1, \dots, U_m)$ in $\mathcal{U}(N)^m$ (m -tuples of unitary $N \times N$ matrices), we define a Hardy space $H^2(\mathbf{U})$ to consist of functions $f: \mathcal{R} \rightarrow \mathbb{C}^N$, holomorphic on $\mathcal{R} \setminus \gamma$, subject to the jump conditions $f(z_-) = U_r f(z_+)$ for $z \in \gamma_r$ for each $r = 1, \dots, m$ (so $\|f(z)\|^2$ is continuous and single-valued on \mathcal{R}), and so that the well-defined integral

$$\|f\|_{H^2(\mathbf{U})}^2 = \int_{\partial \mathcal{R}} \|f(\zeta)\|^2 d\omega_{t_0}$$

is finite. Then the space $H^2(\mathbf{U})$ is a reproducing kernel Hilbert space over \mathcal{R} (with some appropriate convention as to how elements are defined on γ); we denote its $\mathbb{C}^{N \times N}$ -valued reproducing kernel function by $K^{\mathbf{U}}: H^2(\mathbf{U}) = \mathcal{H}(K^{\mathbf{U}})$. These kernels enter into the admissible-kernel formulation of the criterion for the $\mathcal{S}_{\mathcal{R}}(\mathbb{C}^N)$ -interpolation problem to have a solution.

Theorem 4.2. *Suppose that we are given an $N \times N$ matrix-valued function S_0 on the subset \mathcal{R}_0 of \mathcal{R} . Then the following are equivalent:*

- (1) *There is a function S in the Schur class $\mathcal{S}_{\mathcal{R}}(\mathbb{C}^N)$ with $S|_{\mathcal{R}_0} = S_0$.*
- (2) *There is a matrix-valued function $((\mathbf{x}, \mathbf{w}), z) \mapsto H_{\mathbf{x}, \mathbf{w}}(z)$ on $\mathbb{T}_{\mathcal{R}}^N \times \mathcal{R}_0$, bounded and measurable in (\mathbf{x}, \mathbf{w}) for each $z \in \mathcal{R}_0$, together with a finite measure ν on $\mathbb{T}_{\mathcal{R}}^N$ so that*

$$I - S_0(z)S_0(w)^* = \int_{\mathbb{T}_{\mathcal{R}}^N} H_{\mathbf{x}, \mathbf{w}}(z) (I_{\mathcal{X}_{\mathbf{x}, \mathbf{w}}} \otimes (I - S_{\mathbf{x}, \mathbf{w}}(z)S_{\mathbf{x}, \mathbf{w}}(w)^*)) H_{\mathbf{x}, \mathbf{w}}(w)^* d\nu(\mathbf{x}, \mathbf{w})$$

for all $z, w \in \mathcal{R}_0$.

- (3) *For each $\mathbf{U} = (U_1, \dots, U_m)$ in $\mathcal{U}(N)^m$ and for each $Y: \mathcal{R}_0 \rightarrow \mathbb{C}^{N \times N}$, the kernel*

$$k(z, w) := \text{tr} (Y(w)^* (I - S_0(w)^* S_0(z)) Y(z)) K^{\mathbf{U}}(z, w) \quad (4.13)$$

is a positive kernel on \mathcal{R}_0 .

Proof. The equivalence of (1) and (2) is a consequence of Theorem 3.1, once the result of Theorem 4.1 is plugged in.

The equivalence of (1) and (3) is a consequence of Remark 3.7, once it is verified that the set

$$(\Psi_{\mathcal{R}}^N)^0 := \{K^{\mathbf{U}} : \mathbf{U} \in \mathcal{U}(N)^m\} \quad (4.14)$$

is a generating set for the set of admissible kernels $\mathcal{K}_{\Psi_{\mathcal{R}}^N}(\mathbb{C}^N)$. Rather than doing this, we observe that a solution criterion for the $\mathcal{SA}_{\mathcal{R}}(\mathbb{C}^N)$ -interpolation problem was obtained in [17, Theorem 1.5] (as a consequence of the lifting theorem from [14]), but in a somewhat different, more convoluted form than the form (4.13). If one works with right multiplication operators on the space $\mathcal{H}(K^U)_{\mathbb{C}^N}$ rather than with left multiplication operators on a left-side tensoring of the reproducing kernel Hilbert space consisting of row-vector functions as is done in [17], one arrives at the solution criterion (4.13) as presented here. \square

Remark 4.3. We note that the scalar-valued case $N = 1$ of criterion (3) in Theorem 4.2 is due to Abrahamse [1]—note that the extra parameter $Y(z)$ washes out in this case. It was later shown by Ball-Clancey [18] that no open subset of the kernels K^U ($U \in \mathcal{U}(1)^m$) can be omitted for the validity of this result. However, for the case of the annulus, if one takes the set of interpolation nodes \mathcal{R}_0 to be finite and prespecified, then two kernels suffice [54]. While the Abrahamse result extends to the matrix-valued setting for the annulus case (using only scalar-valued kernels), McCullough and Paulsen [39, 40], using the C^* -algebra approach to interpolation theory, showed that the Fedorov-Vinnikov result fails for the matrix-valued case. All this story is reviewed nicely in [26]. We do not address such minimality issues here.

For the case of the annulus ($m = 1$), by using results of McCullough [38] it is possible to obtain a more explicit test-function collection as follows. We take \mathcal{R} to have the concrete form $\mathcal{R} = \mathbb{A}_q$ where

$$\mathbb{A}_q = \{z \in \mathbb{C} : q < |z| < 1\}$$

for a number q satisfying $0 < q < 1$. It is established in [38] that there is a curve $t \mapsto \varphi_t$ of inner functions on \mathbb{A}_q (constructed from the Ahlfors function for \mathbb{A}_q based at the point $\sqrt{q} \in \mathbb{A}_q$) with the following property: for a $(U, t) \in \mathcal{U}(N) \times \mathbb{T}^n$ (where $\mathcal{U}(n)$ denotes the set of $N \times N$ unitary matrices and \mathbb{T}^n is the N -torus $\{t = (t_1, \dots, t_n) : |t_j| = 1 \text{ for } 1 \leq j \leq N\}$), set

$$\Phi_{U,t}(z) = U \begin{bmatrix} \varphi_{t_1}(z) & & \\ & \ddots & \\ & & \varphi_{t_N}(z) \end{bmatrix}$$

and

$$R_{U,t}(z) = (I_N + \Phi_{U,t}(z))(I - \Phi_{U,t}(z))^{-1};$$

then, for each $(\mathbf{x}, \mathbf{w}) \in \mathbb{T}_{\mathbb{A}_q}^N$ there is a choice of invertible $N \times N$ matrix X and a $(U, t) \in \mathcal{U}(N) \times \mathbb{T}^N$ so that

$$F_{\mathbf{x}, \mathbf{w}}(z) + F_{\mathbf{x}, \mathbf{w}}(z)^* = X(R_{U,t}(z) + R_{U,t}(z)^*)X^* \text{ for all } z \in \mathbb{A}_q. \quad (4.15)$$

We are now ready to introduce a new test-function class for $\mathcal{S}_{\mathbb{A}_q}^N$, namely:

$$\tilde{\Psi}_{\mathbb{A}_q}^N = \{\Phi_{U,t} : (U, t) \in \mathcal{U}(N) \times \mathbb{T}^N\}. \quad (4.16)$$

We then have the following result.

Theorem 4.4. *The matrix-valued Schur class over the annulus $\mathcal{S}_{\mathbb{A}_q}(\mathbb{C}^N)$ is identical to the matrix-valued test-function Schur-Agler class $\mathcal{SA}_{\tilde{\Psi}_{\mathbb{A}_q}^N}$ where $\tilde{\Psi}_{\mathbb{A}_q}^N$ is given by (4.16).*

Proof. Suppose first that $S \in \mathcal{SA}_{\tilde{\Psi}_{\mathbb{A}_q}^N}$. Then the right multiplication operator R_S is contractive on $\mathcal{H}(K)_{\mathbb{C}^N}$ for each admissible kernel K in $\mathcal{K}_{\tilde{\Psi}_{\mathbb{A}_q}^N}(\mathbb{C}^N)$. Such kernels include the Fay kernel associated with the Hardy space $H^2(\omega_t) \otimes \mathbb{C}^N$ over \mathbb{A}_q . This observation is enough to conclude that $S \in \mathcal{S}_{\mathbb{A}_q}(\mathbb{C}^N)$.

Conversely suppose that $S \in \mathcal{S}_{\mathbb{A}_q}(\mathbb{C}^N)$. To show that $S \in \mathcal{SA}_{\tilde{\Psi}_{\mathbb{A}_q}^N}(\mathbb{C}^N)$, by Theorem 3.8 it suffices to show: *for any representation $\pi: H_{\tilde{\Psi}_{\mathbb{A}_q}^N}^\infty(\mathbb{C}^N) \rightarrow \mathcal{L}(\mathcal{K})$ such that $\|\pi(\Phi_{U,t})\| \leq 1$ for all $(U,t) \in \mathcal{U}(N) \times \mathbb{T}^N$, it follows that $\|\pi(S)\| \leq 1$.* By replacing π with $r \cdot \pi$ with $r < 1$ and then taking a limit as r tends to 1, without loss of generality we may suppose that $\|\pi(\Phi_{U,t})\| < 1$ for each (U,t) . Then $\pi(R_{U,t}) = (I - \pi(\Phi_{U,t}))^{-1}(I + \pi(\Phi_{U,t}))$ is a well-defined bounded operator on \mathcal{K} such that

$$\pi(R_{U,t}) + \pi(R_{U,t})^* = 2(I - \pi(\Phi_{U,t}))^{-1}(I - \pi(\Phi_{U,t})\pi(\Phi_{U,t})^*)(I - \pi(\Phi_{U,t})^*)^{-1} > 0. \quad (4.17)$$

From (4.15), we see that, for each fixed $(\mathbf{x}, \mathbf{w}) \in \mathbb{T}_{\mathbb{A}_q}^N$, $\pi(F_{\mathbf{x}, \mathbf{w}})$ is a well-defined bounded operator on \mathcal{K} satisfying

$$\pi(F_{\mathbf{x}, \mathbf{w}}) + \pi(F_{\mathbf{x}, \mathbf{w}})^* = (X \otimes I_{\mathcal{K}})(\pi(R_{U,t}) + \pi(R_{U,t})^*)(X^* \otimes I_{\mathcal{K}}). \quad (4.18)$$

From (4.17) we read off that $\pi(F_{\mathbf{x}, \mathbf{w}})$ has positive real part. We next obtain $\pi(S_{\mathbf{x}, \mathbf{w}})$ as a Cayley transform of $\pi(F_{\mathbf{x}, \mathbf{w}})$:

$$\pi(S_{\mathbf{x}, \mathbf{w}}) = (\pi(F_{\mathbf{x}, \mathbf{w}}) + I)^{-1}(\pi(F_{\mathbf{x}, \mathbf{w}}) - I).$$

From the relation

$$I - \pi(S_{\mathbf{x}, \mathbf{w}})\pi(S_{\mathbf{x}, \mathbf{w}})^* = 2(\pi(F_{\mathbf{x}, \mathbf{w}}) + I)^{-1}(\pi(F_{\mathbf{x}, \mathbf{w}}) + \pi(F_{\mathbf{x}, \mathbf{w}})^*)(\pi(F_{\mathbf{x}, \mathbf{w}})^* + I)^{-1}$$

combined with (4.18), we see that $\|\pi(S_{\mathbf{x}, \mathbf{w}})\| \leq 1$. Finally, since $S \in \mathcal{S}_{\mathbb{A}_q}(\mathbb{C}^N)$, S has an Agler decomposition as in (4.11). Applying the hereditary functional calculus with the representation π through this integral representation gives

$$I - \pi(S)\pi(S)^* = \int_{\mathbb{T}_{\mathbb{A}_q}^N} \pi(H_{\mathbf{x}, \mathbf{w}})(I - \pi(S_{\mathbf{x}, \mathbf{w}})\pi(S_{\mathbf{x}, \mathbf{w}})^*)\pi(H_{\mathbf{x}, \mathbf{w}})^* d\nu(\mathbf{x}, \mathbf{w}).$$

Since $\|\pi(S_{\mathbf{x}, \mathbf{w}})\| \leq 1$ for each $(\mathbf{x}, \mathbf{w}) \in \mathbb{T}_{\mathbb{A}_q}^N$, we read off from this last expression that $\|\pi(S)\| \leq 1$ \square

As a corollary of Theorem 4.4 combined with Theorem 3.1, we get the following structure theorem for the Schur-Agler class over the annulus \mathbb{A}_q . To this end we introduce the space $\hat{\mathbb{T}}_{\mathbb{A}_q}^N = (\mathcal{U}(N)/\mathcal{U}(1)^N) \times \mathbb{T}^N$, where here $\mathcal{U}(1)^N$ is identified with unitary diagonal $N \times N$ matrices, and the action of $\mathcal{U}(1)^N$ on $\mathcal{U}(N)$ is given by

$$u: U \mapsto Uu \text{ for } u = \begin{bmatrix} u_1 & & \\ & \ddots & \\ & & u_N \end{bmatrix} \in \mathcal{U}(1)^N.$$

For $([U], t) \in \widehat{\mathbb{T}}_{\mathbb{A}_q}^N$, we abuse notation somewhat and set

$$\Phi_{[U],t} = \Phi_{U,t} U^*.$$

Note that $\Phi_{[U],t}$ is well-defined (independent of the choice of representative of the coset $[U]$). Note that each $\Phi_{[U],t}$ is normalized to satisfy $\Phi_{[U],t}(1) = I_N$ as well as $\Phi_{[U],t}(\sqrt{q}) = 0$. Furthermore the expression $I - \Phi_{U,t}(z)\Phi_{U,t}(w)^*$ is independent of choice of coset representative for $[U]$. Also it is easily checked that the set of admissible kernels \mathcal{K}_Ψ associated with a given collection of test functions Ψ depends on the functions $\psi \in \Psi$ only through the expressions $I - \psi(z)\psi(w)^*$. Hence the result of Theorem 4.4 can equally well be stated as:

$$\mathcal{S}_{\mathbb{A}_q}^N(\mathbb{C}^n) = \mathcal{SA}_{\widehat{\Psi}^N}(\mathbb{C}^N) \quad (4.19)$$

where we have set

$$\widehat{\Psi}_{\mathbb{A}_q}^N = \{\Phi_{[U],t} : ([U], t) \in \widehat{\mathbb{T}}_{\mathbb{A}_q}^N\}.$$

Then the following corollary is an immediate consequence of our main theorem on the test-function Schur-Agler class, namely Theorem 3.1.

Corollary 4.5. *Suppose that $S \in \mathcal{S}_{\mathbb{A}_q}(\mathbb{C}^N)$. Then the following hold:*

- (1) *S has an Agler decomposition of the form*

$$\begin{aligned} I - S(z)S(w)^* \\ = \int_{\widehat{\mathbb{T}}_{\mathbb{A}_q}^N} H_{[U],t}(z) (I_{\mathcal{X}_{[U],t}} \otimes (I - \Phi_{[U],t}(z)\Phi_{[U],t}(w)^*)) H_{[U],t}(w)^* d\nu([U], t). \end{aligned} \quad (4.20)$$

- (2) *There is a representation ρ of $C(\widehat{\mathbb{T}}_{\mathbb{A}_q}^N, \mathcal{L}(\mathbb{C}^N))$ on a Hilbert space \mathcal{X} and a unitary colligation matrix*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathbb{C}^N \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathbb{C}^N \end{bmatrix}$$

so that S has the transfer-function realization

$$S(z) = D + C(I - \rho(\mathbb{E}(z))A)^{-1}\rho(\mathbb{E}(z))B.$$

Remark 4.6. An appealing conjecture is that the Agler decomposition (4.20) is *minimal* in the sense of [29, Section 5.1] and [30, Section 3.6].

4.2. The constrained Schur class over the unit disk. Following [26, 17], we define the *constrained Hardy space* H_1^∞ over the unit disk \mathbb{D} to consist of bounded analytic functions s on \mathbb{D} such that $s'(0) = 0$. One can check that this is still an algebra. In this section we identify a class of test functions Ψ_1^N for which the unit ball $\overline{\mathcal{B}}(H^\infty)^{N \times N}$ of the algebra of $N \times N$ matrices over H_1^∞ (with norm equal to the multiplier norm as multiplication operators on $(H^2)^N$) can best identified as the test-function Schur-Agler class $\mathcal{SA}_{\Psi_1^N}(\mathbb{C}^N)$.

The analysis parallels that of Section 4.1 for the Schur class over a finitely connected planar domain. One first identifies the extreme points for the Herglotz class \mathcal{H}_1^N consisting of $N \times N$ matrix-valued functions F on \mathbb{D} satisfying the normalization $F(0) = I$ together with the side constraint $F'(0) = 0$. Such functions are exactly the Cayley transforms

$$F(z) = (I - S(z))^{-1}(I + S(z))$$

of functions S in the closed unit ball $\overline{\mathcal{B}}(H_1^\infty)^{N \times N}$ of the matrix-valued constrained Hardy algebra $(H_1^\infty)^{N \times N}$ subject to the normalization $S(0) = 0$. As is the case

for any matrix-valued Herglotz function on \mathbb{D} , there is a positive matrix-valued measure μ on \mathbb{T} so that F has the Herglotz representation

$$F(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta).$$

The constraint that $F(0) = I_N$ is equivalent to $\mu(\mathbb{T}) = I_N$; following the terminology used in Section 4.1, we then say that μ is an $N \times N$ *quantum probability measure*. The constraint that $F'(0)$ equals zero (i.e., that $F \in \mathcal{H}_1^N$) imposes the constraints on the measure μ :

$$F'(0) = \int_{\mathbb{T}} \zeta^{-1} d\mu(\zeta) = \int_{\mathbb{T}} \bar{\zeta} d\mu(\zeta) = 0.$$

Taking real and imaginary part then gives us two real constraints

$$\int_{\mathbb{T}} \operatorname{Re} \zeta d\mu(\zeta) = 0, \quad \int_{\mathbb{T}} \operatorname{Im} \zeta d\mu(\zeta) = 0. \quad (4.21)$$

We thus see that the convex set \mathcal{H}_1^N (the constrained matrix-valued Herglotz class over \mathbb{D}) is affinely equivalent to the convex set of measures

$$\mathcal{C}_1^N = \{\mu : \mu = N \times N \text{ quantum probability measure such that (4.21) holds}\}.$$

This convex set of measures is compact in the weak-* topology (viewing complex $N \times N$ matrix-valued measures as the dual space of \mathbb{C}^N -valued continuous functions on \mathbb{T}) and hence, by the Kreĭn-Milman theorem, has extreme points. By the same general results from [20] leading to the identification of the set (4.9) of the normalized Herglotz class $\mathcal{H}(\mathcal{R})_I$ over the planar domain \mathcal{R} , it follows that the extreme points of \mathcal{C}_1^N can be described as follows. We let $\hat{\Theta}^N$ consist of all pairs (\mathbf{t}, \mathbf{w}) where $\mathbf{t} = (t_1, \dots, t_n)$ is an n -tuple of points on the unit circle \mathbb{T} (with $1 \leq n \leq 3N$) and $\mathbf{w} = (W_1, \dots, W_n)$ is an n -tuple of $N \times N$ matrix weights such that the following property holds: $\mathbf{0} = 0 \otimes I_N$ is in the interior of the C^* -convex hull of $\phi(\mathbf{t}) \otimes I_N$, where we set

$$\phi(\mathbf{t}) = \left\{ \begin{bmatrix} \operatorname{Re} t_1 \\ \operatorname{Im} t_1 \end{bmatrix}, \dots, \begin{bmatrix} \operatorname{Re} t_n \\ \operatorname{Im} t_n \end{bmatrix} \right\} \subset \mathbb{R}^2.$$

with a choice of matrix barycentric coordinates of $\mathbf{0}$ with respect to $\phi(\mathbf{t}) \otimes I_N$ equal to $\{W_1, \dots, W_n\}$ (refer back to Section 4.1 for the definition of terms). One consequence of the definitions is that, for any such $(\mathbf{t}, \mathbf{w}) = (t_1, \dots, t_n; W_1, \dots, W_n)$ in $\hat{\Theta}^N$, it holds that

$$\sum_{r=1}^n (\operatorname{Re} t_r) W_r = 0, \quad \sum_{r=1}^n (\operatorname{Im} t_r) W_r = 0. \quad (4.22)$$

Associated with each $(\mathbf{t}, \mathbf{w}) \in \hat{\Theta}^N$ is a holomorphic $N \times N$ -matrix function on the unit disk given by

$$F_{\mathbf{t}, \mathbf{w}}(z) = \sum_{r=1}^n \frac{t_r + z}{t_r - z} W_r.$$

These functions are holomorphic on \mathbb{D} with positive real part, and moreover, as a consequence of (4.22), have the property that $F'_{\mathbf{t}, \mathbf{w}}(0) = 0$. In fact, it can be shown

that the set of all such functions $\{F_{\mathbf{t}, \mathbf{w}} : (\mathbf{t}, \mathbf{w}) \in \mathbb{T}_1^N\}$ is exactly the set of extreme points for the normalized constrained Herglotz class over \mathbb{D} , i.e., the class

$$(\mathcal{H}_1^N)_{I_N} := \{F : \mathbb{D} \rightarrow \mathbb{C}^{N \times N} : F \text{ holomorphic, } \operatorname{Re} F(z) \geq 0 \text{ for } z \in \mathbb{D}, \\ F(0) = I_N, F'(0) = 0\}.$$

By using Choquet theory it then follows that a general element F of $(\mathcal{H}_1^N)_{I_N}$ has an integral representation of the form

$$F(z) = \int_{\hat{\Theta}^N} F_{\mathbf{t}, \mathbf{w}}(z) \, d\nu(\mathbf{t}, \mathbf{w})$$

for some probability measure on $\hat{\Theta}^N$.

We note that $(\mathcal{H}_1^N)_{I_N}$ is exactly the Cayley transform of the normalized constrained Schur class

$$(\mathcal{S}_1^N)_0 = \{S : \mathbb{D} \rightarrow \mathbb{C}^{N \times N} : S \text{ holomorphic, } \|S(z)\| \leq 1 \text{ for } z \in \mathbb{D}, \\ S(0) = 0, S'(0) = 0\},$$

i.e.,

$$S \in (\mathcal{S}_1^N)_0 \Leftrightarrow F := (I - S)^{-1}(I + S) \in (\mathcal{H}_1^N)_{I_N}, \\ F \in (\mathcal{H}_1^N)_{I_N} \Leftrightarrow S := (F + I)^{-1}(F - I) \in (\mathcal{S}_1^N)_0.$$

In particular, for each $(\mathbf{t}, \mathbf{w}) \in \mathbb{T}_1^N$ we may define functions $S_{\mathbf{t}, \mathbf{w}} \in (\mathcal{S}_1^N)_0$ which in turn leads us to the following collection of functions in $(\mathcal{S}_1^N)_0$:

$$\Psi_1^N := \{S_{\mathbf{t}, \mathbf{w}}(z) = (F_{\mathbf{t}, \mathbf{w}}(z) + I)^{-1}(F_{\mathbf{t}, \mathbf{w}}(z) - I) : (\mathbf{t}, \mathbf{w}) \in \hat{\Theta}^N\}. \quad (4.23)$$

Following the proof of Theorem 5.4 in [20] (the parallel result for the matrix Schur class over a planar domain \mathcal{R} in place of \mathcal{S}_1^N) then leads to the integral Agler decomposition for the normalized constrained Schur class: *given $S \in (\mathcal{S}_1^N)_0$ there is a function $((\mathbf{t}, \mathbf{w}), z) \mapsto H_{\mathbf{t}, \mathbf{w}}(z)$ on $\hat{\Theta}^N \times \mathbb{D}$, bounded and measurable in $(\mathbf{t}, \mathbf{w}) \in \mathbb{T}_1^N$ for each fixed z , together with a probability measure ν on $\hat{\Theta}^N$, so that*

$$I - S(z)S(w)^* = \int_{\hat{\Theta}^N} H_{\mathbf{t}, \mathbf{w}}(z) (I - S_{\mathbf{t}, \mathbf{w}}(z)S_{\mathbf{t}, \mathbf{w}}(w)^*) H_{\mathbf{t}, \mathbf{w}}(w)^* \, d\nu(\mathbf{t}, \mathbf{w}). \quad (4.24)$$

If S is in the strict constrained Schur class ($S \in \overline{\mathcal{B}}(H_1^\infty)^{N \times N}$ with $\|S(0)\| < 1$), then there is a choice of matrix Möbius transformation on the $N \times N$ -matrix ball $T_{S(0)}$ so that $T_{S(0)}[S(z)]$ is in the normalized constrained Schur class (\mathcal{S}_1^N) (see e.g. [20, Section 5]). Using this one can see that functions S in the strict but unnormalized Schur class $\mathcal{S}_1^N := \overline{\mathcal{B}}(H_1^\infty)^{N \times N}$ have the continuous Agler decomposition (4.24) as well.

Once this Agler decomposition is in hand, by using the same techniques as used in the proofs of Theorems 4.1 (adaptations to the matrix-valued setting of arguments in [29] and [30]), one can arrive at the following result.

Theorem 4.7. *With $\Psi_1^N \subset (\mathcal{S}_1^N)_0$ given by (4.23), we have the identity*

$$\overline{\mathcal{B}}(H_1^\infty)^{N \times N} = \mathcal{S}\mathcal{A}_{\Psi_1^N}.$$

There is also a dual pair of solution criteria for the interpolation problem for the class \mathcal{S}_1^N . We first need to introduce the generating set of admissible kernels for the class $\mathcal{K}_{\Psi_1^N}(\mathbb{C}^N)$ as follows. For each isometric $2N \times 1$ matrix, written as $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ with

α and β equal to $N \times 1$ column vectors satisfying $\alpha^* \alpha + \beta^* \beta = 1$, we introduce the collection of $N \times N$ -matrix kernel functions

$$(\Psi_1^N)^0 = \{K^{\alpha, \beta}(z, w) := (\alpha z + z\beta)(\alpha^* + \bar{w}\beta^*) + \frac{z^2 \bar{w}^2}{1 - z\bar{w}} I_N : \alpha, \beta \in \mathbb{C}^{N \times 1}, \alpha^* \alpha + \beta^* \beta = 1\}. \quad (4.25)$$

Then we have the following result.

Theorem 4.8. *Suppose that we are given an $N \times N$ matrix-valued function S_0 on the subset \mathbb{D}_0 of the unit disk \mathbb{D} . Then the following are equivalent:*

- (1) *There is a function S in the restricted Schur class \mathcal{S}_1^N with $S|_{\mathbb{D}_0} = S_0$.*
- (2) *There is a matrix-valued function $((\mathbf{t}, \mathbf{w}), z) \mapsto H_{\mathbf{t}, \mathbf{w}}(z)$ on $\hat{\Theta}^N \times \mathbb{D}_0$, bounded and measurable in (\mathbf{t}, \mathbf{w}) for each fixed $z \in \mathbb{D}_0$, together with a finite measure ν on $\hat{\Theta}^N$, so that*

$$I - S_0(z)S_0(w)^* = \int_{\hat{\Theta}^N} H_{\mathbf{t}, \mathbf{w}}(z) (I_{\mathcal{X}_{\mathbf{t}, \mathbf{w}}} \otimes (I - S_{\mathbf{t}, \mathbf{w}}(z)S_{\mathbf{t}, \mathbf{w}}(w)^*)) H_{\mathbf{t}, \mathbf{w}}(w)^* d\nu(\mathbf{t}, \mathbf{w}).$$

- (3) *For each $2N \times 1$ isometric matrix $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and for each $Y: \mathbb{D}_0 \rightarrow \mathbb{C}^{N \times N}$, the kernel*

$$k(z, w) = \text{tr} (Y(w)^* (I - S_0(w)^* S_0(z)) Y(z)) K^{\alpha, \beta}(z, w) \quad (4.26)$$

(where $K^{\alpha, \beta}$ is given by (4.25)) is a positive kernel on \mathbb{D}_0 .

Proof. The proof parallels that of Theorem 4.2. To verify the equivalence of condition (2) with existence of a solution of the interpolation problem, use Theorem 4.7 in combination with Theorem 3.1. By Remark 3.7, the validity of condition (3) follows if we can verify that the collection $(\Psi_1^N)^0$ given by (4.25) is a generating set for the collection of admissible kernels $\mathcal{K}_{\Psi_1^N}(\mathbb{C}^N)$. However, rather than doing this we use Theorem 1.3 from [17]. As was the case for the Schur class over a domain \mathcal{R} , the form presented there is somewhat different from the form (4.26) as presented here. However, one can follow the argument in [17] and work with right multiplication operators on $\mathcal{H}(K^{\alpha, \beta})_{\mathbb{C}^N}$ rather than left multiplication operators on a left-sided tensor of the coefficient space with a reproducing kernel Hilbert space of row-vector functions to arrive at the form (4.26) as the solution criterion. \square

Remark 4.9. As was observed in connection with Corollary 4.5, the Schur-Agler class \mathcal{SA}_Ψ associated with a collection of test functions Ψ depends on the functions $\psi \in \Psi$ only through the kernels $I - \psi(z)\psi(w)^*$. Hence, for $S_{\mathbf{t}, \mathbf{w}}$ in the test-function class Ψ_1^N we may define an equivalence relation $S_{\mathbf{t}, \mathbf{w}} \sim S_{\mathbf{t}', \mathbf{w}'}$ when there is a unitary constant matrix U so that $S_{\mathbf{t}', \mathbf{w}'}(z) = S_{\mathbf{t}, \mathbf{w}}(z)U$. To choose one representative out of each equivalence class, we may normalize $S \in \Psi_1^N$ so that $S(1) = I_N$. This has the effect of restricting the parameter (\mathbf{t}, \mathbf{w}) in $\hat{\Theta}^N$ to those such that 1 is one of the points in the set of points $\mathbf{t} = (1, t_2, \dots, t_n)$ with associated weight W_1 invertible; in this way we get a new smaller parameter space Θ^N . Then we have $\overline{\mathcal{B}}(H_1^\infty)^{N \times N} = \mathcal{SA}_{\tilde{\Psi}_1^N}$ where $\tilde{\Psi}_1^N = \{S_{\mathbf{t}, \mathbf{w}} : (\mathbf{t}, \mathbf{w}) \in \Theta^N\}$ is this restricted class of test functions.

For the case $N = 1$ (the scalar case), Theorem 4.7 is due to Dritschel-Pickering [30]. In this case the parameter space $\hat{\Theta}^1 =: \hat{\Theta}$ can be described in geometric terms as consisting of (1) triples of points on the unit circle such that 0 is in the interior of the associated triangle, with the weights then being the barycentric coordinates

of 0 with respect to this triangle, or (2) a pair of antipodal points on the unit circle with weights then necessarily $(\frac{1}{2}, \frac{1}{2})$. When the reduction described in the previous paragraph is carried out, one restricts to triples of points $\mathbf{t} = (1, t_2, t_3)$ which include 1 and there is only one antipodal pair of points $(1, -1)$. These authors also show that this space Θ with its natural topology is homeomorphic to the unit sphere. They also show that the collection $\tilde{\Psi}_1^1$ is a minimal collection of test functions for $\overline{\mathcal{B}}H_1^\infty$. Whether $\tilde{\Psi}_1^N$ is a minimal collection of test functions for $\overline{\mathcal{B}}(H_1^\infty)^{N \times N}$ in general we leave as an open question.

As we have seen, there is a dual issue of finding minimal generating sets for admissible collections of kernels $\mathcal{K}_\Psi(\mathbb{C}^N)$, as well as finding small generating sets for such $\mathcal{K}_\Psi(\mathbb{C}^N)$. In particular, it would be interesting to see a direct proof that $(\Psi_{\mathcal{R}}^N)^0$ in (4.14) generates $\mathcal{K}_{\Psi_{\mathcal{R}}^N}$ and that the set $(\Psi_1^N)^0$ in (4.25) generates $\mathcal{K}_{\Theta^N}(\mathbb{C}^N)$. We note that the proofs of the interpolation results from [1, 14, 26, 17] use the dual factorization approach (see [25] for a unified setting); an independent proof of the generating property for $(\Psi_{\mathcal{R}}^N)^0$ and $(\Psi_1^N)^0$ would mean that Theorem 3.1 gives an independent proof of these interpolation results.

Remark 4.10. An alternative description of H_1^∞ is $\mathbb{C} + z^2H^\infty$. Many of the results concerning the space H_1^∞ have been generalized to more general algebras of the form $\mathbb{C} + BH^\infty$ where B is a Blaschke product (see e.g. [50]). We believe that the results from [20] are sufficiently flexible to lead to test-function Schur-Agler-class characterizations of matrix-valued versions of these more general algebras as well.

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